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Rapports de Recherche

N° 1384

Programme 2
Calcul symbolique, Programmation et Génie logiciel

HOMOTOPIC DEDUCTIONS IN UNIFICATION LOGIC

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Janvier 1991

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Abstract

Investigations on functional equations in simply typed λ-calculus led to unification logic, an equational logic whose constructivism gives an interpretation of deductions as paths in the hypotheses graphical representation. This geometric semantics modelizes the flow of information in this typed λ-calculus by particle transition systems on graphs, and characterizes deductions by deadlock freeness and connectedness. This characteristic property introduces new surgery tools on deductions, in the first place an homotopy equivalence on deductions. It also defines the sequential and conjugacy classes for deductions of fixed-point equations, whose classification in a type inference context motivated these developments. Here, this taxonomy is realized by canonically associating to a fixed point deduction an element of the first homotopy group of the graph of hypotheses.

Déductions Homotopes dans la Logique de l'Unification

Résumé

Des calculs sur des équations fonctionnelles dans le λ-calcul simplement typé ont établi l'existence d'une logique équationnelle, baptisée logique de l'unification, dont le caractère très constructif fournit une interprétation des déductions par des chemins dans la représentation graphique des hypothèses. Cette sémantique géométrique modélise le flot d'information dans ce λ-calcul par l'intermédiaire de systèmes de transitions sur des graphes, et caractérise les déductions par l'absence de blocage de ces systèmes et la connexité des chemins. Cette propriété caractéristique fournit des outils de classification des déductions, en premier lieu une classification par homotopie. Elle fournit également les classes séquentielles et de conjuguaison pour les déductions d'équations au point-fixe, dont la classification dans un contexte d'inférence des types a motivé ces recherches. Dans le présent article, cette taxonomie est réalisée en associant de façon canonique à une déduction au point fixe un élément du groupe fondamental du graphe des hypothèses.

Homotopic Deductions in Unification Logic *

Philippe LE CHENADEC

Investigations on functional equations in simply typed λ-calculus led to unification logic, an equational logic whose constructivism gives an interpretation of deductions as paths in the hypotheses graphical representation. This geometric semantics modelizes the flow of information in this typed λ-calculus by particle transition systems on graphs, and characterizes deductions by deadlock freeness and connectedness. This characteristic property introduces new surgery tools on deductions, in the first place an homotopy equivalence on deductions. It also defines the sequential and conjugacy classes for deductions of fixed-point equations, whose classification in a type inference context motivated these developments. Here, this taxonomy is realized by canonically associating to a fixed point deduction an element of the first homotopy group of the graph of hypotheses.

Contents

* *1980 Mathematics Subject Classification (1985 Revision).* Primary 03F05, 03F07, 05C10, 05C25, 55Q05; Secondary 03B15, 03B40, 68N15.
Research funded by CEC Basic Research Action 3245, "Logical Frameworks".

1 Introduction

Some properties of the logic of unification, introduced in [35], are presented. Their investigation was motivated by an homotopy classification of fixed-point equations, deducible in this logic LE from some set of equations between first-order terms. In §2,3 we include a synopsis of the formal system LE, a constructive version of usual equational logic, found while working on functional equations in first-order typed λ-calculus. Insights on these equations have potential applications to programming languages, especially for the type inference problem. By the theorem of Kleene, which states that the partial recursive functions are exactly the λ-definable numeric functions [1], this problem may be studied in the framework of typed λ-systems. A type reconstruction procedure fills in missing type information in such a class of algorithms. None is known for large classes, besides the first-order typed λ-calculus, as was first shown by Hindley and Milner [23, 39, 9]. However, this calculus is of poor proof-theoretic strength [1]. We advocate in support of a geometric approach towards this problem for the second-order or polymorphic λ-calculus of Girard and Reynolds [15, 46], which defines, by theorems of Girard, a significant class of algorithms, namely those provably total in second-order Peano arithmetic, which is reputed to include the algorithms of classical analysis [15]. Insights on this problem for Girard's polymorphic system F should also shed some light on the same problem for related systems, such as the Calculus of Constructions of Coquand and Huet [7]. A deduction in the formal system LE, from some set of equations $\mathcal{E}$, possesses a natural semantics in a path algebra $\Gamma(\mathcal{E})$, associated to the standard graphical representation $T(\mathcal{E})$ of the equations $\mathcal{E}$ (see e.g. [12, 8]), which defines a ring of homotopy $\Pi(\mathcal{E})$, including conjugate subgroups isomorphic to the fundamental group $\pi_1(T(\mathcal{E}))$ and the corresponding conjugate group rings isomorphic to $\mathbf{Z}[\pi_1(T(\mathcal{E}))]$ [49, 55, 38]. This semantical representation in both algebras $\Gamma(\mathcal{E})$ and $\Pi(\mathcal{E})$ introduces new tools for deduction surgery and taxonomy, by use of geometrical notions such as coverings and homotopy. The results of the present paper include a first characterization of deduction semantics by deadlock freeness, a second one via homotopy, universal cover lifting and "projective" completion, a characterization of cut-free proofs by a discrete variational principle, the introduction of the homotopy contraction of deductions, of the sequentialization of parallel deductions and of the pruning of a deduction along some occurrence of the terms in its conclusion. This material is applied to fixed-point deductions, thereby classified via the first homotopy group of the hypotheses graph.

In type inference, an algorithm represented as an untyped λ-term should be second-order typed only if it cannot be first-order typed: we have to measure the obstruction to first-order typing. Such a measure is approximated here by a set of fixed-point deduction classes, represented by sequential and homotopy reduced deductions, in normal form with respect to a cut-elimination process [18], defined by a finite, confluent and well-founded set of equations RE [26]. These deductions prove fixed-point equations, consequence in LE of a given set $\mathcal{E}$ of first-order equations. In type inference, the set $\mathcal{E}$ is equal to the set of first-order type equations associated to the given untyped λ-term. The deduction classes can be represented in the first homotopy group of a space, which is the standard topological realization of the graph $T(\mathcal{E})$ of first-order terms, members of equations in $\mathcal{E}$. This first graph defines a unification graph $S(\mathcal{E})$, equal to $T(\mathcal{E})$ quotiented by $\mathcal{E}$, and this representation comes down via the induced morphism in homotopy of the projection onto the topological realization of this last graph. This graph $S(\mathcal{E})$ is the analog of the Cayley graph of a group for the algebra defined by the *relations* $\mathcal{E}$ over the set of variables in $\mathcal{E}$, now considered as a set of *constants*. The topological interpretation is not necessary, but facilitates the comprehension of the paper, which remains elementary and combinatorial throughout. Especially, we intensively use the syntactic tools underlying the Knuth-Bendix and Newman lemmas [41, 31, 26]. A guiding principle is the existence of a correspondence between syntactical notions and morphisms between the graphical representation of the syntactic objects: e.g. to an occurrence of some term M in the term N corresponds an injective morphism $\phi : T(M) \hookrightarrow T(N)$ between their graphical realization.

We now outline this deduction representation. Each deduction $\mathcal{D}$ in LE possesses a unique homotopically reduced contraction. The existence proof of this homotopy contraction, result syntactically established in [35] for a subclass of deductions, requires the equivalence $\Omega(\mathcal{D}) = \Omega(\mathcal{D}')$ iff $\mathcal{D} =_{RE} \mathcal{D}'$, where $\Omega(\mathcal{D})$ is the path representation of $\mathcal{D}$ in $\Gamma(\mathcal{E})$ and $=_{RE}$ is the decidable congruence on deductions defined by the confluent and well-founded set of equations RE of LE. The paths involved in this representation possess a certain amount of parallelism, expressed by a sum operation on the algebra $\Gamma(\mathcal{E})$. This parallelism originates in non-variable first-order terms, whose subterms should be computed synchronously. Such a synchronous parallelism is formalized by the particle system

PA described below. Usual paths of graph theory will be referred to as sequential paths. The interpretation of $\mathcal{D} \vdash^{\mathcal{E}}_{LE} M = N$ has a natural product decomposition $\Omega(\mathcal{D}) = LER$ where the paths L and R record the construction of M and N respectively while the path E, the equality component of $\mathcal{D}$, describes the construction of the proof that M equals N. The subpaths of the path L (resp. R) split up into paths that construct M (resp. N) and equality components of the subdeductions of $\mathcal{D}$ that link together these paths. The deduction $\mathcal{D}$ is sequential when all its equality components, both E and those in L and R, are sum-free. The properties of this subpath partition are quite well reflected under the projective/affine classification, or closed/open paths if one prefers. The introduction and informal use of these notions is motivated by the operational behaviour of deductions, as outlined in §3.3 of [35]. It is also technically justified by the structure of transitions relating elements in a state space $St(\mathcal{E})$ over $T(\mathcal{E})$, that come on stage as follows. In order to establish both the equivalence between the syntactic and the semantic equalities on deductions and the fact that the homotopy contraction of $\Omega(\mathcal{D})$ is of the form $\Omega(\mathcal{D}')$ for some deduction $\mathcal{D}' \vdash^{\mathcal{E}}_{LE} M = N$, we need to characterize deductions semantics among elements of the path algebra $\Gamma(\mathcal{E})$. This result is obtained through a modelization of the message-passing or operational behaviour of deductions, via particle systems, or discrete dynamical systems in the topological interpretation, discrete semiflows more precisely [27]. When the first-order terms in $\mathcal{E}$ are propositional approximations of predicate formulæ, as in Herbrand's theorem [48], this operationality describes the behaviour of the flow of information, constructed from the predicate arguments via the resolution rule of predicate calculus [12]. A particle system is a sequence of states in $St(\mathcal{E})$ that possess a common path $P \in \Gamma(\mathcal{E})$ as support. Pebbles or particles are distributed on the initial state of the path P according to its local structure. They are associated by pairs including a positive and a negative particle. The positive ones follow the local orientation of the deduction, the negative ones the opposed orientation. The orientation is arbitrarily chosen: the flow of information is bidirectional. A path in $\Gamma(\mathcal{E})$ is the semantics of some deduction iff it is deadlock-free, connected and extremal. This last notion comes from the fact that information flows in $T(\mathcal{E})$ from a target vertex to a target vertex (outdegree 0), *through* a source vertex (indegree 0). Without this condition, the local modification of the conveyed information is trivial. The synchronous moves of particles between states of a particle system are formalized by transition rules that define a finite, well-founded transition system PA, confluent on deduction semantics.

The path P is deadlock-free when every particle visits every subpath in P, which is expressed by the condition that every source (resp. target) of the PA-normal form of its initial state possesses every negative (resp. positive) particle. Connectedness corresponds to the familiar topological notion, here formalized via a source-target restriction on paths. Hence a deduction in LE provides a coherent support for a communication. The positive-negative partition of particles corresponds to the notion of positive-negative occurrences of subformulæ in proof theory, which are known to reflect the strength of a subexpression occurrence in a formula [18, 35]. The transition rules of the system PA naturally split into projective and affine rules, which introduces the projective completion of a deduction. Besides witnessing for the adequacy of the affine/projective terminology, this completion simplifies some proofs, which is the classical use of projective space. Now, particle systems are defined on the data-space $T(\mathcal{E})$. But such transition sequences can be projected onto the quotient graph $S(\mathcal{E})$. There, the two particles in a pair projectively move without being separated and the equality components of deductions are null-homotopic, modulo the elements defining the equational relation. The projective rules of PA express on $S(\mathcal{E})$ the condition that projective paths grow by conjugation in $\Pi(S(\mathcal{E}))$ ($\Pi(\mathcal{E})$ being a shorthand for $\Pi(T(\mathcal{E}))$). Further, on this graph, a trajectory can be associated to particles, which is a sequential path. These facts emphasize the idea that a deduction on the data space $T(\mathcal{E})$ should be considered as the support of some computation on the space $S(\mathcal{E})$ presented by this data.

An analysis of the system RE is provided by the particle systems. A projective transition rule of PA, the sum rule, is specifically parallel. It is related to non-variable proper terms of binary inference rules of LE, i.e. the two terms in the rule premisses that are identified. Hence, the number of sum rules in a maximal transition sequence of a deduction path $\Omega(\mathcal{D})$ gives a measure of the parallelism present in $\mathcal{D}$. The introduction of covering graphs allows us to unshare the hypotheses of deductions. A maximal unsharing defines the linearization of $\mathcal{D}$ and its universal covering graph, which discloses a discrete variational property of the system RE. When a potential is assigned to vertices of the path $\Omega(\mathcal{D})$, lifted to this covering graph, the natural parallel particle system associated to $\mathcal{D}$ defines a sum over its states of the potential. In an RE-class, this sum becomes minimum for cut-free deductions. The potential increases between two states under a projective sum transition. A deduction is sequential iff its covering graph is the universal cover of the graph $T(\mathcal{E})$. Besides the parallel transition system of a deduction, there exists a sequential system that gives another property of normal forms. Their sequential system is characterized by the fact that a *maximal* projective component of a path P, once partially travelled, will be completed before passing to another maximal projective component of P. While cut rules have an intrinsic geometric significance, the remaining equations of the set RE merely express a computation of left and right introductions, designed as symmetrically as possible. Covering graphs give another characterization of path semantics. By projective completion, we may assume that such a path is projective. Therefore, when interpreted on a covering graph, which is of the form $T(\mathcal{E})$ for some set of equations $\mathcal{E}$, a path is the denotation of some deduction iff it is connected, extremal and its projection on $S(\mathcal{E})$ is null-homotopic (mod. elements defining the equational relation).

The deduction surgery goes well beyond normal forms and homotopy contractions. These two operations do not necessarily remove the parallelism present in a deduction.

But working out some examples unfolds the existence of a unique sequential deduction associated to fixed-point parallel deductions. The path algebra is non-distributive, and some applications of the distributivity law to $\Omega(\mathcal{D})$, those feasible without leaving the set of deduction denotations, correspond to local sequentializations of parallel binary rules instances in $\mathcal{D}$. If the non-variable proper term of such a rule instance is the first-order term $f(M, N)$, the rule is replaced by two binary rules that successively identify the two occurrences of both M and N. The distribuivity law, hence this operation is non-linear: one of the subdeductions whose conclusion is premiss of the sequentialized binary rule is duplicated. This operation splits up into projective and affine sequentialization, the latter shifts the projective and affine partition of a deadlock-free path. On fixed-point deductions, it does not always preserve the conclusion of $\mathcal{D}$: sequential provability is weaker than provability in LE. In such cases, this sequentialization can only be performed together with a "parallel" conjugation in $\Pi(\mathcal{E})$, corresponding to a cyclic permutation of the hypotheses of $\mathcal{D}$. This apparent complication disappears by observing that the underlying invariant is the homotopy loop in $\pi_1(T(\mathcal{E}))$ associated to a fixed-point deduction. The last operation is the pruning of a deduction along some sequential path of its conclusion. It removes from $\mathcal{D}$ the subdeductions that are not incident to this sequential path. In the rings $\Pi(\mathcal{E})$ and $\Sigma(S(\mathcal{E}))$, the latter being the quotient of $\Pi(S(\mathcal{E}))$ by the subring generated by the elements defining the equational relation, these operations merely correspond to the substraction of subdeductions' semantics. Naturally, not every deduction can be sequentialized or pruned, but a fixed-point deduction can be so reduced, its homotopy interpretation being invariant under surgery. Combining this homotopy and sequential surgery with pruning and conjugation, we have canonically associated to any fixed-point deduction an element of the first homotopy group of the graphical representation of the deduction hypotheses.

Other approaches to polymorphic type inference have been investigated, which are essentially combinatorial, see the papers of Kfoury et al and of Mitchell [29, 40]. Henglein and Mairson have announced a complexity lower-bound for polymorphic type inference [21]. Giannini and Ronchi [14] have given an example of a strongly normalizable λ-term without type in the second-order calculus. Krivine [34] has proven that a simple algorithm given by Maurey could not be expressed in this type discipline with the expected types. The semantics of second-order calculus has been explored by various authors, see the papers by Bruce, Meyer and Mitchell, and by Girard [3, 16]. Introductions to domains of theoretical computer science relevant to the present paper can be found in chapters 6, 7, 8, 12 and 19 of [54], written by N. Dershowitz and J.-P. Jouannaud, H.P. Barendregt, J.C. Mitchell, C.A. Gunter and D.S. Scott, and R. Milner respectively. Pebbling has been used by several authors either in complexity theory to get time-space trade-offs, or in parallelism theory to get deadlock characterizations for various algorithms, see the papers by Hopcroft, Paul, Tarjan and Pippenger [24, 36, 42, 44]. For other relations of pebbling and equational logic see the resource analysis of Kozen [32]. A comprehensive reference to the literature on unification is provided by the volume edited by C. Kirchner [30]. Algorithmic questions on first-order unification have been investigated by Courcelle, Huet, Paterson and Wegman, and by Huet for higher-order unification [8, 43, 25]. Its complexity has been analyzed by Dwork, Kanellakis and Mitchell, and by Lewis and Statman [11, 37]. References to potential and graph theory can be found in [20, 5], and on combinatorial group theory and graphs in [10, 38]. Pruning has already been introduced in natural deduction by Goad [19]. Related work on proof theory of equational calculi can be found in papers by Kreisel and Tait, and Statman [33, 50]. Especially, an ancestor of the system RE can be found in the study by Statman of equational deductions, which failed to be a cut-elimination process by the absence of an elimination rule. Besides, Statman's research has given evidence for the existence of significant mathematics behind functional equations in typed λ-calculus [51, 52]. The semantics of deductions has many similarities with Girard's linear logic [17]. Particularly, the representation algebras can be endowed with several sums and products reflecting a distinction here emphasized under the categories affine/projective. The commutativity of linear logic, especially of the tensorization operator, as well as the central position of the morphism between the hypotheses graph and the space graph prevents the direct use of linear logic for our present goal of homotopy classification. Pebbling and deadlock freeness however is reminiscent of the longtrip characterization of linear logic's proof-nets. Clearly, this part of linear logic, with its criterion of global coherence without short-circuits, is the one that had the most important influence on the present notion of pebbling. Moreover, Girard's leitmotiv that proof-theory is the search for geometrical symmetries behind proofs is here recurrent. The definition of the cut-elimination process of LE has benefited from criticism by G. Huet and the definition of the algebra $\Gamma(\mathcal{E})$ has been strengthened after useful comments by J.-Y. Girard. The existence of a group theoretic model of equational logic was known de facto to geometers, as witnesses the following quotation of E. Cartan: "Si en effet on cherche à préciser la notion d'*égalité*, qui s'introduit dès le début de la Géométrie, on est amené à dire que deux figures sont égales quand on peut passer de l'une à l'autre par une certaine opération géométrique, appelée déplacement" (original emphasis) [4].

I wish to thank Prof. J. Tits for a helpful discussion, as well as Prof. J.L. Krivine for his advice. Professors A.R. Meyer, J.C. Mitchell and D.S. Scott fostered this research by giving audience to early reports of the results. Thanks also to G. Huet and J.Y. Girard, as well as Ph. Flajolet, G. Kahn, B. Lang, and J.-J. Lévy, who provided the scientific environment where this research could be conducted.

2 Definition of Unification Logic

Let $\mathcal{T}(\mathcal{V})$ be the free algebra of first-order terms, for simplicity over one binary function symbol f and over a denumerable set of variables $\mathcal{V}$, and $\mathcal{E}(\mathcal{V})$ be the set of equations between members of $\mathcal{T}(\mathcal{V})$. In the type inference setting, this function symbol becomes the arrow of propositional implication in minimal calculus [12, 18]. Unification is the resolution in $\mathcal{T}(\mathcal{V})$ of equations from $\mathcal{E}(\mathcal{V})$. Under the above restriction, solvability of such equations is equivalent to the absence of non-trivial fixed-point equations, consequence of the given equations in unification logic. With several function symbols, one should add the absence of non-homogeneous equations (i.e. whose members have distinct head function symbols). Also, we analyze deducibility in this logic LE from some fixed set of equations between terms.

2.1 Definitions and Notations

As customary in syntactic studies, we use both occurrences, here binary words that single out subterms, and the context notation. The terminology introduced for surgery on first-order terms will also be used on expressions denoting elements in the path algebras. For the definition of equational graphs below, we remind that an equation $M = N$ is a predicate formula with the equality symbol as predicate symbol. Applied to the term $f(M, N)$, $M, N \in \mathcal{T}(\mathcal{V})$, the empty occurrence ϵ references the whole term, 0 references the subterm M and 1 references N, and so on inductively in both M and N. Similarly, we can define in the obvious way the set $\mathcal{O}(M)$ of occurrences of the term M. A non-trivial occurrence $O = O'1$ (resp. $O'0$) in $\mathcal{O}(M)$ possesses a *complementary* occurrence $\overline{O} = O'0$ (resp. $O'1$). The largest common prefix of the occurrences O_1 and O_2 is noted $O_1 \wedge O_2$. The head symbol of the term $f(M, N)$ is the symbol f, of the equation $M = N$ the equality symbol, while a variable has no head function symbol. We use the notation $C[\_]$, $D[\_]$ $E[\_], \ldots$ for contexts. The *context* $C[\_]$ is a term with one "hole", formally a distinguished variable whose unique occurrence in $C[\_]$ is denoted by O_C. The notation $C[M]$ denotes the term obtained from $C[\_]$ by substituting M for the hole. The empty occurrence defines the trivial context, noted $[\_]$. Two contexts $C[\_]$ and $D[\_]$ are equivalent iff the occurrences of their holes are equal, $O_C = O_D$ (in presence of several function symbols, we further require identity along this common occurrence of the function symbols in both contexts). When working with several holes contexts, we write $C[\_, \ldots, \_]$, where the sequence $\_, \ldots, \_$ enumerates the holes in the leftmost-outermost ordering of occurrences: $O_1 < O_2$ iff either O_1 properly prefixes O_2 or $O_1 = O0O'_1$ and $O_2 = O1O'_2$ for some occurrences O, O'_1 and O'_2. The expression M/O denotes the subterm of M at occurrence O, $O \in \mathcal{O}(M)$. Such a subterm is proper when $O \neq \epsilon$. The syntactic equality (identity) of terms is noted $\equiv$ and the set of variables of some object O is noted $\mathcal{V}(O)$. An equation is strict when its members are variables. The equivalence relation on $\mathcal{V}(\mathcal{E})$ generated by the strict equations from $\mathcal{E}$ is noted $=_s$.

A solution of some set of equations $\mathcal{E}$ is described by a morphism on $\mathcal{T}(\mathcal{V})$ that identifies members of equations in $\mathcal{E}$. Morphisms are defined by their values on variables, and are called *substitutions* following the terminology in use. Assuming the set of variables $\mathcal{V}$ totally ordered by $\leq$, we define the greater lower bound of two terms or substitutions by:

- $x \wedge y = y \wedge x = x$, $x, y \in \mathcal{V}$, $x \leq y$,
- $x \wedge f(M,N) = f(M,N) \wedge x = x$, $x \in \mathcal{V}$, $M, N \in \mathcal{T}(\mathcal{V})$,
- $f(M_1,N_1) \wedge f(M_2,N_2) = f(M_1 \wedge M_2, N_1 \wedge N_2)$, $M_i, N_i \in \mathcal{T}(\mathcal{V})$, $i = 1,2$,
- $(\sigma_1 \wedge \sigma_2)(M) = \sigma_1(M) \wedge \sigma_2(M)$, $M \in \mathcal{T}(\mathcal{V})$.

The morphism $\sigma_1 \wedge \sigma_2$ solves $\mathcal{E}$ when both σ_1 and σ_2 do. Also, $\bigwedge_i \sigma_i$, σ_i a substitution giving one solution to $\mathcal{E}$, describes the most general solution of $\mathcal{E}$ (with respect to $\leq$). A substitution σ will be applied to contexts, holes being fixed by σ, and to sets of terms or equations: $\sigma(\mathcal{U}) = \{\sigma(M) \mid M \in \mathcal{U}\}$, $\sigma(\mathcal{E}) = \{\sigma(M) = \sigma(N) \mid M = N \in \mathcal{E}\}$. The domain of σ is the set of variables $dom(\sigma) = \{x \mid \sigma(x) \not\equiv x\}$. A substitution σ is *geometric* iff $\sigma(\mathcal{V}) \subseteq \mathcal{V}$, and is a *permutation* iff σ restricted to $\mathcal{V}$ is one-one onto $\mathcal{V}$, equivalently σ is an automorphism of $\mathcal{T}(\mathcal{V})$. The identity substitution is noted Id. An *instance* of some term M is a term of the form $\sigma(M)$ for some morphism σ.

A binary relation $\rightarrow$ defined on a set S is *confluent* iff for all $x, y, z \in S$ such that $z \rightarrow x$ and $z \rightarrow y$ there exists an element $t \in S$ with $x \rightarrow t$ and $y \rightarrow t$. It is well-founded or normalizing iff it satisfies the descending chain condition. When $x \rightarrow y$, y is said to be *derived* from x. In applications, the relation $\rightarrow$ is the reflexive-transitive closure of a one-step relation $\Rightarrow$, which is itself generated from a set of equations in a first-order formalism, also called rules in the present context, by replacements in some term of left-hand side instances of some equation by its associated right-hand side instances. An occurrence of a left-hand side instance in a term M is called a *redex* of M. The redex is *contracted* in the term N iff N is equal to M where the redex has been replaced by the corresponding instance of the right-hand side. A set of equations is said to be confluent and normalizing when its associated relation $\rightarrow$ is. By the Knuth-Bendix and Newman lemmas, when such a relation is well-founded and confluent each element possesses a unique *normal form*, see [41, 31, 26]. A derivation sequence is a finite or infinite sequence $(x_i)_i$ such that $x_i \Rightarrow x_{i+1}$. This terminology will be applied to both sets of equations RE below and PA in §4, as well as to the homotopy, sequential and pruning surgery on deductions.

The following notations and definitions concerning graphs are largely borrowed from [53, 47, 10]. A graph G is defined by the data $(V, E, \partial^-, \partial^+)$, where V is the vertex set, E the edge set, $\partial^-, \partial^+ : E \rightarrow V$ the incidence functions, respectively the *source* and *target* of an edge. The sets V and E are assumed to be countable. The *indegree* (resp. *outdegree*) of a vertex $v \in V$ is the cardinal of the set of edges $e \in E$ incident to v, i.e. such that $\partial^+(e) = v$ (resp. $\partial^-(e) = v$). A *source* (resp. *target*) of G is a vertex with null indegree (resp. outdegree). A vertex is *internal* iff both its indegree and outdegree are positive. A vertex is *shared* iff it has indegree greater than 1, and *unshared* otherwise. The *degree* of a vertex is equal to its indegree plus its outdegree. A leaf of a tree is a vertex whose degree is equal to 1. A graph is locally finite iff every vertex has finite degree. A *chain* is a graph whose vertices are the integers z such that $a \leq z \leq b$, for $a \in \{-\infty\} \cup \mathbf{Z}$ and $b \in \mathbf{Z} \cup \{+\infty\}$, $a \leq b$, and whose edges are the pairs $e_i = (i, i+1)$, $a \leq i < b$, with $\{\partial^-(e_i), \partial^+(e_i)\} = \{i, i+1\}$. A chain is either finite, simply infinite or doubly infinite whenever a, b are integers, $a = -\infty$ or $b = +\infty$, or $a = -\infty$ and $b = +\infty$. Its length is equal to $b - a$ when both a and b are integers or is $+\infty$ otherwise. We note $\mathrm{Ch}(a, b)$ the chain defined by a, b as above and the edges e_i such that $(\partial^-(e_i), \partial^+(e_i)) = (i, i+1)$, $a \leq i < b$.

Let $G_1 = (V_1, E_1, \partial_1^-, \partial_1^+)$ and $G_2 = (V_2, E_2, \partial_2^-, \partial_2^+)$ be two graphs. A graph morphism $\phi : G_1 \rightarrow G_2$ maps vertices to vertices, edges to edges and is orientation-preserving: $\partial_2^- \circ \phi = \phi \circ \partial_1^-$ and $\partial_2^+ \circ \phi = \phi \circ \partial_1^+$. The graph G_1 is a *subgraph* of G_2 iff $V_1 \subseteq V_2$, $E_1 \subseteq E_2$ and the inclusion map of these sets is a morphism. The subgraph G_1 is *full* iff every edge $e \in E_2$ with $\partial_2^-(e), \partial_2^+(e) \in V_1$ belongs to E_1. Any chain is isomorphic to a chain with either $a = 0$, $b = n$, $0 \leq n < +\infty$, or $a = 0$, $b = +\infty$, or $a = -\infty$, $b = 0$, or $a = -\infty$, $b = +\infty$. Such a chain is called standard. A *path* of a graph $G = (V, E, \partial^-, \partial^+)$ is a morphism ϕ from a standard chain to G. This definition will be extended in §3. The paths just defined are *sequential*. The length of the path ϕ is the length of its defining chain, noted $|\phi|$. When the chain is finite, its *source* (resp. *target*) is the vertex $\phi(0)$ (resp. $\phi(n)$). A finite path ϕ is called a loop whenever $\phi(0) = \phi(n)$. A directed path is a morphism from one of the standard chains $\mathrm{Ch}(-\infty, +\infty)$, $\mathrm{Ch}(-\infty, 0)$, $\mathrm{Ch}(0, +\infty)$ or $\mathrm{Ch}(0, n)$, $0 \leq n < +\infty$. A circuit is a directed loop. In a circuit-free graph G, the vertex u is a *successor* of the vertex v iff there exists an edge e such that $\partial^-(e) = u$, $\partial^+(e) = v$, u is then a *predecessor* of v. More generally, when G is circuit-free, the vertex u is a descendant (resp. antecedent) of the vertex v iff there exists a directed path with source u and target v. The graph $G \downarrow u$ (resp. $G \uparrow u$) is the full subgraph of G containing the descendants (resp. antecedents) of u. The vertex u is a half-turn of the path ϕ iff there exists $i > 0$ such that $\phi(i) = u$, $\phi(e_{i-1}) = \phi(e_i)$. To any finite path is naturally associated a unique finite path without half-turns. The distance $d(u, v)$ between two vertices u and v is the minimal length of paths with source u and target v, or is $+\infty$ when no such path exists. A path of minimal length between u and v has no half-turns. The graph G is *connected* when $d(u, v)$ is finite for all $u, v \in V$. A connected component of the graph G is a maximal connected subgraph of G. In type inference, graphs are connected. The graph G is a *tree* iff, for all $u, v \in V$, there exists a unique finite path without half-turn with source u and target v. In a tree we have $d(\partial^-(e), v) = d(\partial^+(e), v) \pm 1$ for any edge e and vertex v. When $d(\partial^-(e), v) = d(\partial^+(e), v) + 1$ (resp. -1), we say that the edge e is directed towards (resp. backwards) the vertex v. A tree is rooted iff it possesses either a unique target and its sources coincide with its leaves, or a unique source and its targets coincide with its leaves. In the former case every edge is directed towards the target, the graph is called a target tree. Besides its target, the vertices of a target tree have outdegree equal to 1 (resp. source tree).

The graph G is a 2-graph iff vertices have outdegree equal to 0 or 2 and the edges leaving an outdegree 2 vertex are ordered, e.g. they are labelled by 0 (left edge) and 1 (right edge). This defines the left (resp. right) successor of such a vertex, and any edge $e \in E$ possesses a *complementary* edge $\overline{e}$. The label of the edge $e \in E$ is noted $l(e)$. A triangular graph is a 2-graph whose internal vertices are unshared. In a 2-graph, a finite directed path p is specified by a triple (v, O, v'), v its source, v' its target, and O an occurrence, concatenation of the labels along its edges. When working with target or source trees and with triangular graphs, the subgraph image of a directed path will sometimes be called a branch. More generally, in an equational framework with a countable operator domain Σ, the definition of a 2-graph generalizes to a Σ-graph, the label $l(e)$ of an edge e being a pair (f, i) with $f \in \Sigma$, $0 \leq i \leq n-1$, where n is the arity of f.

Let G be a 2-graph without infinite directed path. Equivalently, the vertex set of G, with the ancestor relation, is a partially ordered set with both ascending and descending chain conditions. Such graphs are called 2-pographs. We partition vertices into sources of G and non-sources. Similarly we distinguish edges whose source is a source of G, called source edges, from the remaining ones. A morphism of 2-graphs is a label-preserving morphism of graphs. A *source morphism* between 2-graphs is a 2-graph morphism mapping sources to sources. The *unification graph* $U(G)$ is the quotient of G by the smallest, downward closed, equivalence generated by the sources of G, while the *sharing graph* $P(G)$ is the quotient of G by the smallest, upward closed, equivalence generated by the targets of G. The *space graph* $S(G)$ is the quotient of G by the smallest equivalence relation containing the two previous relations. They are compatible with the 2-graph structure: for an equivalence $\simeq$, besides the graph compatibility condition: $e_1 \simeq e_2$ implies $\partial^{\pm}(e_1) \simeq \partial^{\pm}(e_2)$, for edges e_1, e_2, they also verify the 2-graph compatibility condition: for vertices v_1, v_2, $v_1 \simeq v_2$ implies that for all edges e_1, e_2 with $l(e_1) = l(e_2)$, $\partial^-(e_1) = v_1$, $\partial^-(e_2) = v_2$, we have $e_1 \simeq e_2$. We define the source equivalence as the smallest equivalence $\sim$ such that: on vertices of G, $v_1 \sim v_2$ when both vertices v_1 and v_2 have a source of G as common predecessor, or when there exists two edges e_1 and e_2 such that $e_1 \sim e_2$ and $v_i = \partial^+(e_i)$, $i = 1, 2$; and on edges of G, $e_1 \sim e_2$ when $l(e_1) = l(e_2)$ and $\partial^-(e_1) \sim \partial^-(e_2)$. The target equivalence $\approx$ is the smallest equivalence such that: $v_1 \approx v_2$ when both vertices v_1 and v_2 have outdegree 2, are either both sources or not, and their left outgoing edges e_1, e_2 as well as their right ones $\overline{e_1}$, $\overline{e_2}$ are equivalent, $e_1 \approx e_2$ and $\overline{e_1} \approx \overline{e_2}$; and such that two edges e_1 and e_2 are equivalent, $e_1 \approx e_2$, when they are either both sources edges or not, $l(e_1) = l(e_2)$, and their targets are equivalent, $\partial^+(e_1) \approx \partial^+(e_2)$, as well as the targets of their complementary edges, $\partial^+(\overline{e_1}) \approx \partial^+(\overline{e_2})$.

The relation $\cong$ is the smallest equivalence relation closed under the defining clauses of both the source and the target equivalences. Hence, it contains both the source and the target relations. The definition of these relations is well-founded, by the poset condition on the graph G. The 2-graph $U(G)$ is equal to $G/\sim$, $P(G)$ to $G/\approx$, and $S(G)$ to $G/\cong$. The canonical projections define source morphisms, especially important here is the projection $\pi(G) : G \to S(G)$. The definition of $S(G)$ can be extended to an arbitrary 2-graph G, via the definition of the equivalences as least fixed-point of the monotone operators on the complete lattice of equivalences of the graph G defined by the above defining clauses for $\sim$ and $\approx$, in which case we have $S^2 = S$ (such fixed-points are congruences). Let $\phi : G_1 \to G_2$ be a morphism of 2-graphs, and o_1, o_2 be either two vertices or two edges of G_1, then $o_1 \approx o_2$ implies $\phi(o_1) \approx \phi(o_2)$. If ϕ is a source morphism, we further have $o_1 \sim o_2$ implies $\phi(o_1) \sim \phi(o_2)$, and ϕ defines a morphism $S(\phi) : S(G_1) \to S(G_2)$ such that $S(\phi) \circ \pi(G_1) = \pi(G_2) \circ \phi$. The three maps U, S and P are functors from the category of 2-graphs with source morphisms to itself.

2.2 The Logical System LE

The language of the logical system LE is composed of the set $\mathcal{T}(\mathcal{V})$ of first-order terms, together with equations between terms. It does not possess any axiom and its inference rules will be presented in natural deduction style [45]. A *derivation tree* D from a set of equations $\mathcal{E}$ in the system LE is a non-empty finite target tree whose vertices have indegree equal to 1 or 2, and subject to the following conditions. Any two distinct edges with common target are ordered in a left edge and a right one. To any vertex is associated an equation and to any edge an occurrence. The equation associated to a source of D belongs to $\mathcal{E}$. The *conclusion* of D is the equation associated to its target. The minimal subgraph of D containing the edges sharing a common target is an *instance* of some inference rule of LE. The convenient notation of natural deduction exhibits the *premisses* of an inference rule above an horizontal bar, and its *conclusion* below the bar. Accordingly, an informal presentation of the inference rules of the system LE is:

$$LE \left\{ \begin{array}{llll} S\,\dfrac{M = N}{N = M} & E\,\dfrac{C[M] = D[N]}{M = N}\; O_C = O_D & & T\,\dfrac{M = N \quad N = O}{M = O} \\[2ex] IL\,\dfrac{M = N \quad C[N] = O}{C[M] = O} & & & IR\,\dfrac{M = C[N] \quad N = O}{M = C[O]} \end{array} \right.$$

In all rules, the contexts are assumed to be non-trivial. Formally, an instance of the rule IL is a target tree $T = (\{v_1, v_2, v_3\}, \{e_1, e_2\}, \partial^-, \partial^+)$ such that $\partial^-(e_1) = v_1$, $\partial^-(e_2) = v_2$, $\partial^+(e_1) = \partial^+(e_2) = v_3$, and e_1 is the left edge. The empty occurrence labels e_1 while a non-trivial occurrence O_2 labels e_2. An equation $M = N$ labels v_1 and an equation of the form $C[N] = O$, with $O_C = O_2$, is associated to v_2 and the equation $C[M] = O$ to v_3. Instances of the other rules are similarly defined. A *deduction* is an isomorphism class of derivation trees. Two such trees are isomorphic when their underlying trees are isomorphic as graphs under an isomorphism which preserves both the label structure and the edge ordering. By a slight abuse of language, the graph vocabulary will be applied to deductions.

We use the graphical notation of natural deduction to display deductions, e.g. if $\mathcal{D}_1$ is a deduction with conclusion $M = N$ (resp. $\mathcal{D}_2$ and $N = O$), then we can build the deduction $\mathcal{D}$ with conclusion $M = O$, with an instance of the transitivity inference rule:

$$\begin{array}{c} \mathcal{D}_1 \\ M = N \end{array} \qquad\qquad \begin{array}{c} \mathcal{D}_2 \\ N = O \end{array} \qquad\qquad T\,\dfrac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_2 \\ M = N & N = O \end{array}}{M = O}$$

Besides symmetry S and transitivity T, we have an elimination rule E and a left (resp. right) introduction rule IL (resp. IR). The elimination rule states that in order to solve or unify the equation $f(M, N) = f(M', N')$, we should unify both $M = M'$ and $N = N'$. The other rules are more or less standard up to their strong constructivity and atomicity. The term N in the premisses of a binary inference rule instance is its proper term (resp. proper context(s)). The set $\mathcal{W}(\mathcal{D})$ of proper variables of a deduction $\mathcal{D}$ is $\bigcup_N \mathcal{V}(N)$, N ranging over proper terms in $\mathcal{D}$. A deduction is *sequential* iff its proper terms are variables. The system LE_s is the subsystem of LE defined by the same restriction: proper terms are variables. The subsystem LE_s is *sequential*, if we agree that *parallelism* is related to the computations of the left and right subterms M and N of a non-variable term $f(M, N)$. This parallelism is formalized by the transition system PA of §4, and its geometric properties are investigated. See [35] for a detailed presentation and motivation of the system LE. A principal premiss of a unary rule instance is its single premiss and of a binary rule is: any premiss for rule T, the left (resp. right) premiss for rule IR (resp. IL). A directed path of a deduction is a *principal branch* iff all equations associated to its vertices are principal premisses, with the possible exception of the conclusion of the deduction itself. The equation of the source (resp. target) is the hypothesis (resp. conclusion) of the principal branch. An *hypothesis* of $\mathcal{D}$ is an equation associated to a source of $\mathcal{D}$. The set of hypotheses of the deduction $\mathcal{D}$ is noted $\mathcal{A}(\mathcal{D})$. This defines the non-empty set of *principal* hypotheses of a deduction. In proof theory, natural deduction style, a *cut* in a deduction is an introduction instance whose conclusion is premiss of an instance of an associated elimination [18]. We will see below that contracting a cut may increase the number of principal branches. Also, this last notion is intrinsic only for cut-free deductions. Following the accepted logical terminology, the notation $\mathcal{D} \vdash_L^{\mathcal{E}} F$ refers to a deduction $\mathcal{D}$, defined by the formal system L, of the formula F, from some set of formulæ $\mathcal{E}$. For the system LE, the set of deductions defined by $\mathcal{E}$ is noted $\mathrm{Th}_0(\mathcal{E})$. If σ is a substitution and $\mathcal{D}$ a deduction, we define $\sigma(\mathcal{D})$ as the labelled target tree obtained by applying σ to the equations labelling the vertices of $\mathcal{D}$. If $\mathcal{D} \vdash_{LE}^{\mathcal{E}} M = N$, then $\sigma(\mathcal{D})$ is a deduction $\sigma(\mathcal{D}) \vdash_{LE}^{\sigma(\mathcal{E})} \sigma(M) = \sigma(N)$. We have $\sigma(\mathrm{Th}_0(\mathcal{E})) \subseteq \mathrm{Th}_0(\sigma(\mathcal{E}))$. We define a set of *trivial* deductions $\mathrm{Th}_t(\mathcal{E}) = \{\mathcal{D}_x | x \in \mathcal{V}(\mathcal{E})\}$. By convention, their target tree is the empty tree, the set $\mathcal{A}(\mathcal{D}_x)$ is empty while the conclusion of $\mathcal{D}_x$ is the equation $x = x$. Let $\mathrm{Th}(\mathcal{E}) = \mathrm{Th}_0(\mathcal{E}) \cup \mathrm{Th}_t(\mathcal{E})$. A term M is *$\mathcal{E}$-constructible* iff there exists some deduction $\mathcal{D} \vdash_{LE}^{\mathcal{E}} M = M$ in $\mathrm{Th}(\mathcal{E})$. Usually, by a deduction we understand a non-trivial deduction, the trivial ones being naturally associated to the homotopy contractions of §6.

A subdeduction of $\mathcal{D}$ is a subgraph of the form $\mathcal{D} \uparrow v$, v some vertex of $\mathcal{D}$, together with the inherited labelling structure. A subdeduction is itself a deduction. A salient feature of the system LE is its atomicity: to every variable occurrence in the conclusion of an inference is associated a unique occurrence of this variable in one of the premisses. More generally, a subobject, either subterm or subcontext, of a member of the conclusion of some binary rule instance is either split between the two premisses or entirely contained in some member of one of the premisses. A subobject of the conclusion of a deduction belongs to some hypothesis iff it is not split along the path from this hypothesis to the conclusion. The model theory of the system LE explores this property.

3 Semantics of Deductions

The completeness results of the equational logic LE are presented, and the definitions of the representation algebras are given. These constructions differ in two ways from the ususal ones. Completeness is defined via graph immersions instead of validity in all models. The algebras may include elements associated to non-connected geometric paths and are non-distributive.

3.1 Model Theory

To a set of equations $\mathcal{E}$ are associated two *equational graphs* related by a projection $\pi : T(\mathcal{E}) \rightarrow S(\mathcal{E})$, and intermediate equational graphs, among them $U(\mathcal{E})$ and $P(\mathcal{E})$, the unification and the sharing graph. An equational graph G is a pair of a 2-graph $(V, E, \partial^-, \partial^+)$ and a map that associates to any vertex $v \in V$ a non-empty set $\mathrm{O}_G(v)$ of objects, either of terms, $\mathrm{O}_G(v) \subseteq \mathcal{T}(\mathcal{V})$, or of equations, $\mathrm{O}_G(v) \subseteq \mathcal{E}(\mathcal{V})$. The vertex v has outdegree 2 iff there exists a non-variable term $f(M,N)$ or an equation $M = N$ in $\mathrm{O}_G(v)$, in which case M (resp. N) belongs to $\mathrm{O}_G(v')$, v' the left (resp. right) successor of v. If $\mathrm{O}_G(v) \subseteq \mathcal{E}(\mathcal{V})$, the vertex v is necessarily a source of G. Conversely, we require that all sources of G are equational, and that $\mathrm{O}_G(s_1) \cap \mathrm{O}_G(s_2) = \emptyset$ when s_1 and s_2 are distinct sources of G. We also require that a variable x belongs to at most one set $\mathrm{O}_G(v)$, if so the vertex v is then denoted by $V_G(x)$. This requirement is minimal with respect to data sharing issues in computer science. Finally, for any term M occurring in some set $\mathrm{O}_G(v)$, we require that there exists at least one source s of G and a directed path (s, O, v) such that $\mathrm{O}_G(s)$ contains an equation $C[M] = N$ with $O = 0O_C$ (resp. $N = C[M]$ with $O = 1O_C$). Let e be an edge in E. A term $M \in \mathrm{O}_G(\partial^+(e))$ occurs by e iff there exists some term $f(M,N) \in \mathrm{O}_G(\partial^-(e))$ and $l(e) = 0$ (resp. $f(N,M)$ and $l(e) = 1$). The 2-graph $(V, E, \partial^-, \partial^+)$ is the *abstract graph* associated to the equational graph G. A term graph G is defined similarly, with $\mathrm{O}_G(v) \subseteq \mathcal{T}(\mathcal{V})$ for every vertex v of G, and the source conditions removed.

A morphism $\phi : G_1 \rightarrow G_2$ between the equational graphs G_1 and G_2 is a pair of

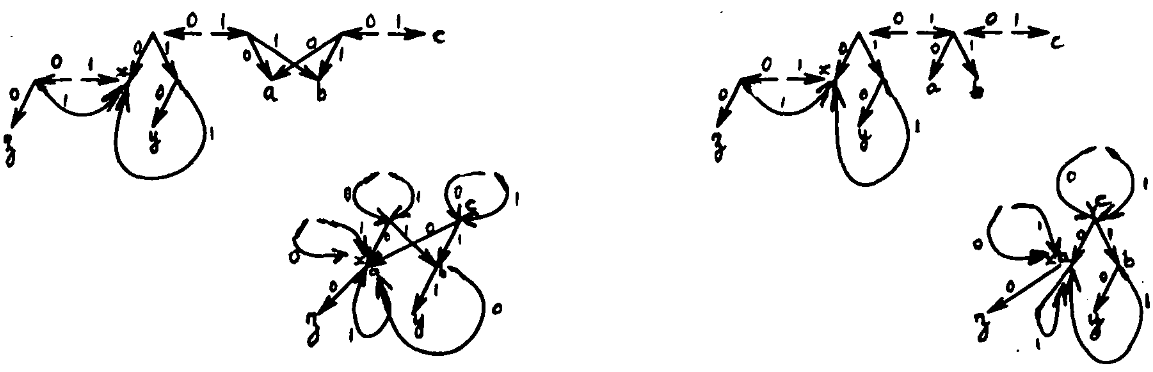

Figure 1: The graphs $T(\mathcal{E}_1)$, $U(\mathcal{E}_1)$, $P(\mathcal{E}_1)$ and $S(\mathcal{E}_1)$.

a source morphism ψ between the abstract graphs of G_1 and G_2, and a substitution σ such that $\sigma(\mathrm{O}_{G_1}(v)) \subseteq \mathrm{O}_{G_2}(\psi(v))$ for any vertex v of G_1. Similarly, morphisms of term graphs are such pairs with ψ a 2-graph morphism. A morphism is *strict* when $\sigma = Id$, *conservative* when $\sigma(\mathrm{O}_{G_1}(v)) = \mathrm{O}_{G_2}(\psi(v))$, and *geometric* iff the substitution σ is geometric (i.e. $\sigma(\mathcal{V}) \subseteq \mathcal{V}$). The morphism $\phi = (\psi, \sigma)$ is an isomorphism of equational graphs iff both ψ and σ are isomorphisms. The requirement on $\mathrm{O}_G(v)$, $v \in V$, introduces a partition between term and equational vertices and edges, an edge e being equational when its source $\partial^-(e)$ is equational, and is a term edge otherwise. Similarly, a variable vertex is a vertex v such that $\mathrm{O}_G(v) \cap \mathcal{V} \neq \emptyset$, and a variable edge has a variable vertex for target. Subscripts of the vertex sets functions will usually be omitted.

Up to 2-graph isomorphism, a term M has a natural representation by a term graph $T(M)$, which is a circuit-free triangular graph with a unique source v such that $\mathrm{O}(v) = \{M\}$ and for any vertex u of $T(M)$, we have $|\mathrm{O}(u)| = 1$. Up to 2-graph isomorphism, the equational graph $T(\mathcal{E})$ is the graphical representation of the set of equations $\mathcal{E}$, built up from graphs isomorphic to $T(M)$, M member of an equation from $\mathcal{E}$. The relation between $\mathcal{E}$ and $T(\mathcal{E})$ is given by a bijective map $\rho_{\mathcal{E}}$ from $\mathcal{E}$ to sources of $T(\mathcal{E})$. The empty graph represents the empty set, and:

- if $\mathcal{E} = \{M_1 = M_2\}$, the graph $T(\mathcal{E})$ has one source v such that $\mathrm{O}(v) = \{M_1 = M_2\}$, $\rho_{\mathcal{E}}(M_1 = M_2) = v$, and if v_1 (resp. v_2) is the left (resp. right) successor of v, the 2-graph $G \downarrow v_i$ is isomorphic to the 2-graph of $T(M_i)$, $i = 1, 2$, and inherits its term structure;

- if $\mathcal{E}$ is the disjoint union $\mathcal{E}' \sqcup \{M_1 = M_2\}$, $T(\mathcal{E})$ is the equational graph that possess two full abstract subgraphs with disjoint edge sets isomorphic to the abstract graph of $T(\mathcal{E}')$ and of $T(\{M_1 = M_2\})$, with the inherited equational structure, $\rho_{\mathcal{E}} \upharpoonright \mathcal{E}' = \rho_{\mathcal{E}'}$ and $\rho_{\mathcal{E}}(M_1 = M_2)$ is the source of the isomorphic copy of $T(\{M_1 = M_2\})$.

For drawings, cf. Fig. 1, with $\mathcal{E}_1 = \{x = f(z, x),\ f(x, f(y, x)) = f(a, b),\ f(a, b) = c\}$. To every *occurrence* of a term M in $\mathcal{E}$ is associated an injective morphism $T(M) \hookrightarrow T(\mathcal{E})$, abusively noted $\psi^{\mathcal{E}}_M$. If G_1 is an equational graph, G_2 a 2-graph and $\phi : G_1 \to G_2$ is an onto source morphism, there exists a minimum equational structure on G_2 defined by $\mathrm{O}_{G_2}(v) = \bigcup_{\phi(v')=v} \mathrm{O}_{G_1}(v')$, with strict morphism $\psi = (\phi, Id)$. Endowed with this equational structure, the abstract graphs $U(T(\mathcal{E}))$, $P(T(\mathcal{E}))$ and $S(T(\mathcal{E}))$ define equational graphs $U(\mathcal{E})$, $P(\mathcal{E})$ and $S(\mathcal{E})$ respectively. The canonical projection $\pi(T(\mathcal{E})) : T(\mathcal{E}) \to S(\mathcal{E})$ is a strict morphism of equational graphs that plays a central rôle in our investigations. By definition and as will be clear below, the graph $S(\mathcal{E})$ is mathematically well-behaved, but not $U(\mathcal{E})$, while the latter is sufficient for computational purposes.

The abstract 2-graphs underlying equational graphs of the form $T(\mathcal{E})$ for some $\mathcal{E}$ are characterized as *triangular 2-pographs whose connected components have at least two vertices*. For in such a case, to any edge or non-target vertex is associated a unique source, hence a unique equation in $\mathcal{E}$. Given such a graph G, this allows the definition of a set $\mathcal{E}_G$ of equations, such that G is isomorphic to $T(\mathcal{E}_G)$ as 2-graphs, $\mathcal{E}_G$ being unique up to isomorphism, provided that we require $|\mathrm{O}_G(v)| = 1$ for every vertex v. If the triangularity requirement is dropped, the pograph still defines a minimum equational structure, where non-variable terms can be shared. Such graphs range between $T(\mathcal{E})$ and $P(\mathcal{E})$, which are both circuit-free. Notice that the graph $U(\mathcal{E})$ is not circuit-free in general. Equational graphs and their morphisms define our working category $\mathfrak{K}(\mathcal{V})$. Let $G_\perp = T(\mathcal{E}(\mathcal{V}))$ and $S_\perp$ be the *collapsed* 2-graph $S(G_\perp)$. The graph $G_\perp$ is a terminal object in the subcategory $\mathfrak{T}(\mathcal{V})$ of $\mathfrak{K}(\mathcal{V})$ whose objects are the equational triangular 2-pographs and morphisms are strict, while the graph $S_\perp$ is terminal in the subcategory of $\mathfrak{K}(\mathcal{V})$ whose morphisms are strict, as well as in the subcategory $\mathfrak{S}(\mathcal{V})$ whose objects are the equational graphs of the form $S(G)$, $G \in \mathfrak{K}(\mathcal{V})$, and morphisms are strict. In categories $\mathfrak{S}(\mathcal{V})$ and $\mathfrak{T}(\mathcal{V})$, by the strictness requirement, the homsets are either empty or singletons. For a graph $G \in \mathfrak{K}(\mathcal{V})$, let $T(G)$ be the graph $T(\mathcal{E}_G)$, where $\mathcal{E}_G = \sqcup_s \mathrm{O}_G(s)$, s a source of G. Similarly, $\rho(G) : T(G) \to G$ and $T(\phi)$ for $\phi : G_1 \to G_2$ are easily defined from the equational structure of the involved graphs. Both $S : \mathfrak{K}(\mathcal{V}) \to \mathfrak{S}(\mathcal{V})$ and $T : \mathfrak{K}(\mathcal{V}) \to \mathfrak{T}(\mathcal{V})$ are covariant functors. We have $T^2 = T$, $S^2 = S$, and $\pi \restriction \mathfrak{T}(\mathcal{V}) = \rho \restriction \mathfrak{S}(\mathcal{V}) : T \to S$ is a natural transformation, $T(\rho(G))$ and $S(\pi(G))$ are the identity morphisms. Further, we have $S \circ T = S$ and $T \circ S = T$, and the restrictions $T \restriction \mathfrak{S}(\mathcal{V})$ and $S \restriction \mathfrak{T}(\mathcal{V})$ are adjoint functors. Notice the complementary feature of S and T, the former is geometrical, the latter equational. The following proposition summarizes standard facts from unification theory.

Proposition 3.1 [43, 11, 8, 35] *Unification is P-complete, hence decidable. The set $\mathcal{E}$ is unifiable iff the unification graph $U(\mathcal{E})$ is both circuit-free and homogeneous. The unification graph $U(\mathcal{E})$ is circuit-free iff the space graph $S(\mathcal{E})$ is circuit-free.*

Given some term M, there exists at most one 2-graph morphism ϕ between the abstract graphs of $T(M)$ and $S(\mathcal{E})$, such that $\phi(V_{T(M)}(x)) = V_{S(\mathcal{E})}(x)$, $x \in \mathcal{V}(M)$. This morphism is noted $\phi^{\mathcal{E}}_M$ if it exists, in which case $V_S(M)$ denotes the image in $S(\mathcal{E})$ of the source of

M. Two terms M and N are equal modulo $\mathcal{E}$, $\models_S^{\mathcal{E}} M = N$, iff the two morphisms $\phi_M^{\mathcal{E}}$ and $\phi_N^{\mathcal{E}}$ exist and $V_S(M) = V_S(N)$, resp. $\models_U^{\mathcal{E}}$ and $U(\mathcal{E})$.

Theorem 3.2 [35] *Let $\mathcal{E} \subseteq \mathcal{E}(\mathcal{V})$ be some set of equations.*
Sequential Completeness of LE_s: *Let (v, O, v') be some finite directed path in $U(\mathcal{E})$, then for all variables $x \in \mathrm{O}_{U(\mathcal{E})}(v)$ and $y \in \mathrm{O}_{U(\mathcal{E})}(v')$, there exists a context $C[\_]$ and deduction $\mathcal{D} \vdash_{LE_s}^{\mathcal{E}} x = C[y]$ with $O_C = O$. Conversely, for any deduction $\mathcal{D} \vdash_{LE_s}^{\mathcal{E}} x = C[y]$, there exists a finite directed path $(V_U(x), O_C, V_U(y))$ in $U(\mathcal{E})$.*
Completeness of LE: $\vdash_{LE}^{\mathcal{E}} M = N$ *iff* $\models_S^{\mathcal{E}} M = N$.

Notice that the theorem can be relativized to 2-pographs. An essential component in the proof of sequential completeness is provided by the following proposition that describes the relation between the geometrical and equational local structure of the equational graph $U(\mathcal{E})$. Completeness follows quite easily from sequential completeness.

Proposition 3.3 [35] *Let $\mathcal{E}$ be a set of equations. In the equational graph $U(\mathcal{E})$, we have:*

1. *For all edges e such that $\mathrm{O}(\partial^+(e)) \cap \mathcal{V}(\mathcal{E}) \neq \emptyset$, there exists a variable $x \in \mathrm{O}(\partial^+(e))$ that occurs by e.*

2. *For all vertices v and pairs (e, e') of distinct edges with $\partial^+(e) = \partial^+(e') = v$, there exists a sequence of pairs (x_j, y_j) of variables in $\mathrm{O}(v)$, $j = 0, \ldots, n-1$, and a sequence of edges e_k, $\partial^+(e_k) = v$, $k = 0, \ldots, n$, with $e_0 = e$, $e_n = e'$, such that $x_j =_s y_j$, x_j occurs by e_j and y_j occurs by e_{j+1}.*

The decidability of LE follows from theorem 3.2. This does not contradict the undecidability of equational logic, which contains the word problem [48]: the logic LE is fully constructive. The subsystem LE_s is sound but incomplete: $\models_U^{\mathcal{E}} M = N$ does not imply $\vdash_{LE_s}^{\mathcal{E}} M = N$, cf. $\mathcal{E} = \{u = f(f(a, b), c),\ u = f(x, f(y, z))\}$ and the equation $u = f(x, c)$.

3.2 Path Algebras

Let $\mathcal{E}$ be some set of equations. The model theory of the system LE reduces validity to the existence and properties of morphisms $\phi_M^{\mathcal{E}} : T(M) \to S(\mathcal{E})$. Besides, there exists a proof theory, based on the path algebra $\Gamma(G)$ of an abstract 2-graph $G = (V, E, \partial^-, \partial^+)$, and related rings: a path ring $\Delta(G)$ and an homotopy ring $\Pi(G)$, and a second homotopy ring $\Sigma(G)$ when $G \in \mathfrak{S}(\mathcal{V})$, as well as the algebras $\Gamma(S(G))$, $\Delta(S(G))$, $\Pi(S(G))$, and $\Sigma(S(G))$. For convenience, $\Gamma(T(\mathcal{E}))$ (resp. $\Delta(T(\mathcal{E}))$...) will be noted $\Gamma(\mathcal{E})$ (resp. $\Delta(\mathcal{E})$...). This proof theory examines the existence of morphisms $\phi_M^{\mathcal{E}}$ via the projection $\pi : T(\mathcal{E}) \to S(\mathcal{E})$. A path is an element of $\Gamma(G)$. By contrast with the paths defined in §2.1, these elements possess a certain amount of parallelism due to the existence of non-unary function symbols in first-order terms. This is expressed by a sum operation on $\Gamma(G)$, while a product corresponds to path concatenation of graph theory. Finally, the product is fiberwise local:

a product PQ of two paths P and Q in $\Gamma(G)$ will be non-null iff at least one target of P and at least one source of Q lie over the same vertex of the graph $S(G)$.

For a 2-graph $G = (V, E, \partial^-, \partial^+)$, we give a combinatorial definition of the algebra $\Gamma(G)$, as a quotient of a free algebra $A(G)$. Let $S(G) = (V_S, E_S, \partial_S^-, \partial_S^+)$, $\pi : G \to S(G)$, and $\mathsf{N}[B]$ be the free additive monoid on the set B, endowed with a join operator defined by $(\sum_B n_b b) \wedge (\sum_B m_b b) = \sum_B \min(n_b, m_b) b$. Let $\mathfrak{P}(V)$ be the power set of V and $b : \mathsf{N}[V] \to \mathfrak{P}(V)$ the morphism of abelian monoids, addition in $\mathfrak{P}(V)$ being the set union (resp. $b_S : \mathsf{N}[V_S] \to \mathfrak{P}(V_S)$). By abuse of notation, we note $\pi : \mathsf{N}[V] \to \mathsf{N}[V_S]$ the morphism defined by $\pi(\sum_V n_v v) = \sum_V n_v \pi(v)$.

The algebra $A(G)$ is finitely generated, with generator set equal to the disjoint union $\mathcal{G}_G \sqcup \mathcal{G}_G{}^{-1} \sqcup \mathcal{I}_G$, where $\mathcal{G}_G$ contains an element per non-source edge of G, and one element per pair of complementary source edges. The set $\mathcal{G}_G^{-1}$ is a set of formal inverses of the elements in $\mathcal{G}_G$, and the set $\mathcal{I}_G = \{e_v \mid v \in V\}$ is isomorphic to the vertex set V of G. The product is associative, the sum is associative and commutative, and possesses a zero element 0, absorbing with respect to the product. The product is *non-distributive* with respect to the sum. The class of these non-distributive algebras is a non-trivial equational variety, and the free algebra $A(G)$ so described exists [6, Cor. 3.3. §IV.3]. An element in $A(G)$ is *trivial* iff it belongs to the subalgebra generated by the elements from $\mathcal{I}_G$. We speak of source, target generators, etc. The label of a non-source generator is the label of its associated edge, and e.g. equal to 0 on source generators and on $\mathcal{I}_G$.

The congruence defining $\Gamma(G)$ as a quotient of $A(G)$ is defined via boundary linear maps $m(\_, u, v) : A(G) \to \mathsf{N}[V]$, $u, v \in V$, that count the number of sequential paths embedded in members of $A(G)$, with source u and target v and whose direct image in the set of sequential paths of $S(G)$ is connected. These maps are defined by:

- $m(e_u, u, u) = 1$, $m(e_u, v, w) = 0$ when v or w is distinct from u, $e_u \in \mathcal{I}_G$;
- $m(a, u, v) = 1$ if either $\partial^-(e) = u$, $\partial^+(e) = v$, where $e \in E$ is the non-source edge associated to $a \in \mathcal{G}_G$, or a is associated to the pair (α, β) of complementary source edges with $l(\alpha) = 0$, and $\partial^-(\alpha) = u$, $\partial^+(\beta) = v$, $m(a, u, v) = 0$ otherwise;
- $m(a^{-1}, u, v) = m(a, v, u)$, $a \in \mathcal{G}_G^{-1}$;
- $m(PQ, u, v) = \sum\limits_{\pi(x)=\pi(y)} m(P, u, x) m(Q, y, v)$, $P, Q \in A(G)$.

These maps are well-defined on the algebra $A(G)$, being compatible with its defining laws. For any $u \in V$ and $P \in A(G)$, let $m^+(P, u) = \sum_{v \in V} m(P, v, u)$ and $m^-(P, u) = \sum_{v \in V} m(P, u, v)$. Both maps $m^\pm(\_, u)$ are linear and enjoy the following properties:

- $m^\pm(e_u, v) = \delta_{u,v}$ (Kronecker symbol), $e_u \in \mathcal{I}_G$;
- $m^\pm(a, v) = \delta_{u,v}$, where $u = \partial^\pm(a)$, $a \in \mathcal{G}_G \cup \mathcal{G}_G{}^{-1}$;

- $m^-(PQ,u) = \sum_{\pi(x)=\pi(y)} m(P,u,x)m^-(Q,y)$, $P,Q \in A(G)$;

- $m^+(PQ,u) = \sum_{\pi(x)=\pi(y)} m^+(P,x)m(Q,y,u)$, $P,Q \in A(G)$.

Similarly, we define four linear maps $\partial^\pm(P) = \sum_{u \in V} m^\pm(P,u)u$ and $\partial_S^\pm = \pi \circ \partial^\pm$. We have $\ker(\partial^-) = \ker(\partial^+) = \ker(\partial_S^\pm)$. Moreover, we endow the algebras $A(G)$ with a linear involution, defined by $\overline{\overline{A}} = A$, $\overline{AB} = \overline{B}\,\overline{A}$, corresponding to inverses on $\mathcal{G}_G \cup \mathcal{G}_G{}^{-1}$ and fixing the elements in $\mathcal{I}_G$. Also, trivial elements are fixed-points of the involution. It will define inverses in the first homotopy group of G, through its conjugate natural embeddings in the homotopy rings obtained from $\Gamma(G)$. Also, this involution will sometimes be noted as an inverse operator. Boundary maps are compatible with the involution: $m(\overline{P},u,v) = m(P,v,u)$, $m^\pm(\overline{P},u) = m^\mp(P,u)$ and $\partial^\pm(\overline{P}) = \partial^\mp(P)$ (resp. $\partial_S^\pm$). Let $\epsilon : \mathbf{N}[V] \to \mathbf{N}$ be the morphism defined by $\epsilon(\sum_{v \in V} n_v v) = \sum_{v \in V} n_v$, we have $\epsilon \circ \partial^+ = \epsilon \circ \partial^-$. For a subset W of some set U, let $p_W : \mathbf{N}[U] \to \mathbf{N}[U]$ be the projection defined by $p_W(\sum_{v \in U} n_v v) = \sum_{v \in W} n_v v$. Given $W \subseteq V_S$, we introduce two linear restriction maps $P \restriction^\pm W$, $P \in A(G)$:

- $e_u \restriction^\pm W = \begin{cases} e_u, & \text{if } \pi(u) \in W; \\ 0, & \text{otherwise;} \end{cases}$

- $a \restriction^- W = \begin{cases} a, & \text{if } b_S(\partial_S^-(a)) \subseteq W; \\ 0, & \text{otherwise;} \end{cases} \quad a \in \mathcal{G}_G \sqcup \mathcal{G}_G{}^{-1};$

- $a \restriction^+ W = \begin{cases} a, & \text{if } b_S(\partial_S^+(a)) \subseteq W; \\ 0, & \text{otherwise;} \end{cases} \quad a \in \mathcal{G}_G \sqcup \mathcal{G}_G{}^{-1};$

- $(PQ) \restriction^+ W = P(Q \restriction^+ W)$ and $(PQ) \restriction^- W = (P \restriction^- W)Q$.

They are well-defined and have the following properties:

- $\partial^\pm(P \restriction^\pm W) = p_W(\partial^\pm(P))$;

- $\partial^-(P \restriction^+ W) = \sum_{\substack{u \in V \\ \pi(v) \in W}} m(P,u,v)u \quad$ (resp. ∂^+, $\restriction^-$);

- $(P \restriction^\pm W') \restriction^\pm W'' = P \restriction^\pm (W' \cap W'') = (P \restriction^\pm W'') \restriction^\pm W'$;

- $(P \restriction^\pm W') \restriction^\mp W'' = (P \restriction^\mp W'') \restriction^\pm W'$;

- $\overline{P \restriction^\pm W} = \overline{P} \restriction^\mp W$.

The algebra $\Gamma(G)$ is the quotient of $A(G)$ by the smallest congruence such that:

- $e^2{}_v \sim e_v$, $e_v \in \mathcal{I}_G$;

- $P \sim 0$ if $\partial^\pm(P) = 0$;

- $e_v P \sim P$ if $b(\partial^-(P)) = \{v\}$, $Pe_v \sim P$ if $b(\partial^+(P)) = \{v\}$;

- $P_1Q_1 \sim P_2Q_2$ if $P_1 \restriction^+ (b_S \circ \partial_S^-)(Q_1) \sim P_2 \restriction^+ (b_S \circ \partial_S^-)(Q_2)$ and $Q_1 \restriction^- (b_S \circ \partial_S^+)(P_1) \sim Q_2 \restriction^- (b_S \circ \partial_S^+)(P_2)$.

When $P \sim Q$, we have $m(P,u,v) = m(Q,u,v)$, $u,v \in V$, hence $m^{\pm}(P,u) = m^{\pm}(Q,u)$ and $\partial^{\pm}(P) = \partial^{\pm}(Q)$. We also have $\overline{P} \sim \overline{Q}$ and $P \restriction^{\pm} W \sim Q \restriction^{\pm} W$, $W \subseteq V_S$. Especially, the algebra $\Gamma(G)$ is trivial iff G is the empty graph. These algebras are involutive and possess linear restriction and boundary maps, still noted by overlining, by $\restriction^{\pm}$ and by $\partial^{\pm}$, with $\ker(\partial^{\pm}) = \{0\}$. Moreover, in $\Gamma(G)$, $PQ = 0$ iff $\partial_S^+(P) \wedge \partial_S^-(Q) = 0$, which establishes that the involution is non-degenerate: $P\overline{P} = 0$ implies $P = 0$. The subalgebra of trivial paths in $\Gamma(G)$ is defined as the image of the algebra of trivial paths of $A(G)$ under the canonical projection. On these paths, we have $\partial_S^-(P) = \partial_S^+(P)$, and $m(P,u,v) > 0$ implies $\pi(u) = \pi(v)$, $u,v \in V$. Members of $\mathcal{I}_G$ are mapped in $\Gamma(S(G))$ onto idempotent elements generating the trivial paths. These idempotents are primitive and e_u and e_v define orthogonal idempotents iff $\pi(u) \neq \pi(v)$ [2]. The algebra $\Gamma(G)$ is cancellative: $A + B = C + B$ implies $A = C$.

Let $\pi : G_1 \to G_2$ be a morphism of graphs, not necessarily orientation preserving. The above construction still defines an algebra $\Gamma(G_1)$. Especially, the identity map of G_2 defines the algebra $\Gamma(G_2)$ and we want to define from π a morphism between $\Gamma(G_1)$ and $\Gamma(G_2)$, giving a natural map from $\Gamma(G)$ to $\Gamma(S(G))$, G a 2-graph. Also, an element $x \in \Gamma(G)$, G a graph, is an atom iff it is non-trivial and does not have non-trivial sum or product decompositions, i.e. $x = y + z$ implies $y = 0$ or $z = 0$, and $x = yz$ implies y idempotent and $z = x$, or z idempotent and $y = x$. The involution maps atoms to atoms without fixed-points. On such path algebras, atoms are in bijective correspondence with generators from $\mathcal{G}_G \sqcup \mathcal{G}_G{}^{-1}$: they are generated by their idempotents and their atoms. Also, we can define a morphism $\pi_\sharp : \Gamma(G) \to \Gamma(S(G))$ by $\pi_\sharp(e_u) = e_v$, if $v = \pi(u)$, and $\pi_\sharp(a)$ is the generator of $\Gamma(S(G))$ associated to $\pi(e)$ if $a \in \mathcal{G}_G$ is associated to the edge e of G, and $\pi_\sharp(a^{-1}) = \pi_\sharp(a)^{-1}$. We have $\pi_\sharp(\overline{A}) = \overline{\pi_\sharp(A)}$, the morphism $\pi_\sharp$ is surjective with trivial kernel $\ker(\pi_\sharp) = \{0\}$. Two distinct idempotents of $\Gamma(S(G))$ are orthogonal.

Conversely, we consider two algebras Γ_i, $i = 1,2$, both having a countable minimal generating set, which are additive monoids and multiplicative semigroups, the zero element being absorbent, and endowed with a linear involution such that $\overline{PQ} = \overline{Q}\,\overline{P}$, $P,Q \in \Gamma_i$, $i = 1,2$. They are related by a morphism $\pi : \Gamma_1 \to \Gamma_2$. We assume that π is surjective and has trivial kernel, that idempotents are fixed by the involution, and that in Γ_2 two distinct idempotents are orthogonal. An element x in Γ_i, $i = 1,2$, is atomic iff it is non-null and $x = y + z$ implies $y = 0$ or $z = 0$, and $x = yz$ implies y idempotent and $z = x$, or z idempotent and $y = x$. We require that idempotents are atomic, hence primitive. An element x is an atom iff it is atomic, non-trivial and there exists two idempotents e', e'' such that $e'x = x$ and $xe'' = x$. We assume that atomic elements split up into idempotents and atoms. Now, the involution necessarily maps atoms to atoms. We further assume that atoms are not fixed-points of the involution. Their set is therefore the disjoint union of a set E_i and its involutory image $\overline{E_i}$, $i = 1, 2$. We assume that there exists well-defined linear maps $m_1(\_, e_1, e_2) : \Gamma_1 \to \mathbb{N}[\mathcal{I}_1]$, for every pair (e_1, e_2) of idempotents in $\mathcal{I}_1$, where $\mathcal{I}_1$ is the set of idempotents of Γ_1. These maps being such that:

- if e is idempotent, $m_1(e, e_1, e_2) = \begin{cases} 1, & \text{if } e = e_1 = e_2; \\ 0, & \text{otherwise;} \end{cases}$
- if a is an atom, $m_1(a, e_1, e_2) = \begin{cases} 1, & \text{if } e_1 a = a e_2 = a; \\ 0, & \text{otherwise;} \end{cases}$
- $m_1(PQ, e_1, e_2) = \sum_{\substack{\pi(x)=\pi(y) \\ x,y \in \mathcal{I}_1}} m_1(P, e_1, x) m_1(Q, y, e_2)$, $P, Q \in \Gamma_1$.

And similarly there exists maps m_2 for Γ_2, the multiplicative condition being replaced by $m_2(PQ, e_1, e_2) = \sum_x m_2(P, e_1, x) m_2(Q, x, v)$, $P, Q \in \Gamma_2$, $e_1, e_2, x \in \mathcal{I}_2$, $\mathcal{I}_2$ the idempotent set of Γ_2. We assume that $m_2 \circ \pi = \pi \circ m_1$ and that π maps idempotents (resp. atoms) onto idempotents (resp. atoms). This implies the existence of boundary maps $\partial_i^{\pm}$, and restriction maps $\restriction_i^{\pm}$, $i = 1, 2$, on both algebras, with $\pi \circ \partial_1^{\pm} = \partial_2^{\pm} \circ \pi$. And we finally assume that:

- $e^2 = e$, $e \in \mathcal{I}_1$,
- $P = 0$ iff $\partial_1^{\pm}(P) = 0$,
- $eP = P$ if $b_1(\partial_1^-(P)) = \{e\}$, $Pe_v = P$ if $b_1(\partial_1^+(P)) = \{v\}$;
- $P_1 Q_1 = P_2 Q_2$ if $P_1 \restriction_1^+ (b_2 \circ \partial_2^- \circ \pi)(Q_1) = P_2 \restriction_1^+ (b_2 \circ \partial_2^- \circ \pi)(Q_2)$ and $Q_1 \restriction_1^- (b_2 \circ \partial_2^+ \circ \pi)(P_1) = Q_2 \restriction_1^- (b_2 \circ \partial_2^+ \circ \pi)(P_2)$;

where $b_i : \mathbb{N}[\mathcal{I}_i] \to \mathfrak{P}(\mathcal{I}_i)$; and that these are the only identities valid in Γ_1, besides the associativity, commutativity and absorption identities (resp. for Γ_2, with π the identity map and $\restriction_1^{\pm}$ being replaced by $\restriction_2^{\pm}$). The algebra Γ_1 is quotient of the free algebra on a minimal generating set for Γ_1, in the variety of these non-distributive algebras with idempotent set $\mathcal{I}_1$ and idempotency laws. By theorem IV.3.11 of [6], this generating set is isomorphic to the set of atoms of Γ_1, for otherwise a non-atom generator would provide an identity valid in Γ_1 besides the above ones. Therefore, Γ_i is isomorphic to $\Gamma(G_i)$, where $G_i = (\mathcal{I}_i, E_i \sqcup \overline{E_i}, \partial_i^-, \partial_i^+)$, $i = 1, 2$. The graphs G_i are unique up to isomorphism, not necessarily orientation preserving. The restriction of the map π to atomic elements defines a graph morphism of G_1 onto G_2. Let G be a graph, up to a subdivision of an edge, so that arity two vertices exist which can become equational sources, G can be endowed with an equational structure. Simple examples establish that not all maps $\pi : \Gamma_1 \to \Gamma_2$ can be associated to Σ-graphs source morphisms, Σ an operator domain.

The ring $\Delta(G)$ is the ring of additive quotients of the semiring equal to the quotient of $\Gamma(G)$ by distributivity: cf. [2, Ch. II, §7] or [28, p.11], the proofs remains valid even if the semiring is non-commutative, provided that the additive monoid is cancellative. This last monoid must however be commutative and multiplication is given by $(P,Q)(R,S) = (PR+QS, PS+QR)$, the order of operands in products being significant. The natural morphism $f_G : \Gamma(G) \to \Delta(G)$ decomposes in an onto morphism $p_G : \Gamma(G) \to B$ and an injective one $B \hookrightarrow \Delta(G)$, where B is the additive monoid $\Gamma(G)$ quotiented by distributivity, this monoid B is cancellative [2, loc. cit.]. The homotopy ring $\Pi(G)$ is the quotient of $\Delta(G)$ by the congruence generated by the cancellation identities $aa^{-1} = e_u$, a an atom, $u = \partial^-(a)$. When $G \in \mathfrak{S}(\mathcal{V})$, we further define the ring $\Sigma(G)$, quotient of $\Pi(G)$ by the congruence generated by the equations $\alpha = e_u$, α a source atom, $u = \partial^\pm(\alpha) = \partial^\pm_S(\alpha)$. We note $g_G : \Delta(G) \to \Pi(G)$ and $h_G : \Pi(G) \to \Sigma(G)$ the natural projections. Due to the presence of zero divisors in $\Gamma(G)$, and by definition of the algebra $\Gamma(G)$, a source morphism $\phi : G_1 \to G_2$ induces a "direct image" morphism $\Gamma(\phi) : \Gamma(G_1) \to \Gamma(G_2)$, noted $\phi_\sharp$, iff the map $S(\phi)$ is injective (as a set-theoretic map, not as a morphism in $\mathfrak{S}(\mathcal{V})$) [49, 55]. Hence, the maps Γ, Δ, Π and Σ are functors from the category whose objects are morphisms $\pi : G \to S(G)$ and arrows are 2-graph morphisms $\phi : G_1 \to G_2$ such that $S(\phi)$ is injective. The morphism $\Gamma(\phi) = \phi_\sharp$ is then defined by $\phi_\sharp(e_u) = e_v$, if $v = \phi(u)$, $\phi_\sharp(a)$ is the atom of $\Gamma(G_2)$ associated to $\phi(e)$, if the atom a is associated to the edge e of G_1 (resp. $\Delta(\phi) = \phi_\natural$, $\Pi(\phi) = \phi_\star$, $\Sigma(\phi) = \phi_\flat$, with $\phi_\natural(-A) = -\phi_\natural(A)$, $\phi_\star(-A) = -\phi_\star(A)$ and $\phi_\flat(-A) = -\phi_\flat(A)$). These maps are well-defined as the atoms plus the idempotents generate these algebras. Extending the involution on the rings by linearity, $\overline{-P} = -\overline{P}$, we have $\phi_\sharp(\overline{A}) = \overline{\phi_\sharp(A)}$ (resp. $\phi_\natural$, $\phi_\star$, $\phi_\flat$). Direct image morphisms commute with the projection morphisms g_G, h_G, and with f_G. They are injective when their defining graph morphism is. If $\pi : G \to S(G)$ is the natural projection, $S(\pi)$ is injective, being the identity map. Also the morphism $\pi_\sharp$ is well-defined and surjective. We note θ_G the morphism $\pi_\flat \circ h_G \circ g_G \circ f_G = h_{S(G)} \circ g_{S(G)} \circ f_{S(G)} \circ \pi_\sharp : \Gamma(G) \to \Sigma(S(G))$. If $G \in \mathfrak{S}(\mathcal{V})$, the ring $\Delta(G)$ is unitary, with unit $1 = \sum_{v \in G} e_v$ decomposed into a sum of primitive orthogonal idempotents [2, Ch. 8]. In general, the rings $\Delta(G)$ and $\Pi(G)$ are neither nœtherian nor artinian. In the ring $\Delta(G)$, both the restriction and boundary mappings can be extended by linearity, the restriction $\upharpoonright^- W$ corresponds to the projection on the sum of ideals $e_u\Delta(G)$, $\pi(u) \in W$. While no longer atomic, we still call atom the image in the rings of an atom of a path algebra.

Let G be a 2-graph. The *connected* paths of the algebra $\Gamma(G)$ are:

- the idempotents and the atoms;
- the sums $P_1 + P_2$ when both P_1 and P_2 are connected;
- $P = P_1P_2$, $i = 1, 2$, when both P_1 and P_2 are connected and $b(\partial^+(P_1)) = b(\partial^-(P_2))$.

The strongly connected paths are defined like connected paths, with the additional condition in the last clause above that the set $b(\partial^+(P_1)) = b(\partial^-(P_2))$ is singleton. To a path we associate various sets of subpaths:

- $Sub(0) = \emptyset$, $Sub(a) = \{e_{\partial^-(a)}, e_{\partial^+(a)}, a\}$ if a is atomic;
- $Sub(P_1 + P_2) = Sub(P_1) \cup Sub(P_2) \cup \{c_1 + c_2 \mid c_i \in Sub(P_i)\}$;
- $Sub(P_1P_2) = Sub(P_1 \restriction^+ ((b_S \circ \partial_S^-)(P_2))) \cup Sub(P_2 \restriction^- ((b_S \circ \partial_S^+)(P_1))) \cup \{c_1c_2 \mid c_i \in Sub(P_i),\ c_1c_2 \neq 0\}$.

Similarly, we define the set of sequential subpaths $Sub_s(P)$, $P \in \Gamma(G)$, by replacing the last two clauses above by:

- $Sub_s(P_1 + P_2) = Sub_s(P_1) \cup Sub_s(P_2)$;
- $Sub_s(P_1P_2) = Sub_s(P_1 \restriction^+ ((b_S \circ \partial_S^-)(P_2))) \cup Sub_s(P_2 \restriction^- ((b_S \circ \partial_S^+)(P_1))) \cup \{c_1c_2 \mid c_i \in Sub_s(P_i),\ c_1c_2 \neq 0\}$.

Finally, the maximal sequential subpaths of some path are defined by:

- $Sub_{s,m}(0) = \emptyset$, $Sub_{s,m}(a) = \{a\}$ if a is atomic;
- $Sub_{s,m}(P_1 + P_2) = Sub_{s,m}(P_1) \cup Sub_{s,m}(P_2)$;
- $Sub_{s,m}(P_1P_2) = \{c_1c_2 \mid c_i \in Sub_{s,m}(P_i),\ c_1c_2 \neq 0\}$.

The path P is called a *source* path iff $b(\partial^-(P))$ is singleton, and a *target* path iff $b(\partial^+(P))$ is singleton. A 1-1 path is both a source and a target path. A path is *sequential* iff it belongs to the multiplicative semigroup $\Psi(G) \subset \Gamma(G)$ generated by atomic elements. Nonzero sequential paths are 1-1. A 1-1 path is *closed* iff $b(\partial^-(P)) = b(\partial^+(P))$. Connected sequential closed paths are in bijective correspondence with loops. They are partitionned into semigroups $\Psi(G, v)$ by the condition $\partial^\pm(P) = \{v\}$, v a vertex of G. For every path $P \in \Gamma(G)$, its direct image $\pi_\sharp(P) \in \Gamma(S(G))$ is connected. Assume now G connected. If v is any vertex of G, the set $\pi(G, v) = (h_G \circ f_G)(\Psi(G, v)) \subset \Pi(G)$ of equivalence classes of sequential closed paths with source and target v, supplied with the path concatenation, inverse and unit, is a group isomorphic to the fundamental group $\pi_1(G)$, which is a free group [49, 47]. Two such subgroups $\pi(G, v)$ and $\pi(G, v')$ are conjugate. Similarly the group ring $\mathbf{Z}[\pi_1(G)]$ has conjugate natural embeddings in $\Pi(G)$, defined as the subrings $\Pi(G, v)$ generated by $\pi(G, v)$.

We conclude this section by giving two matrix representations of both $\Gamma(G)$ and $\Gamma(S(G))$, G a 2-graph. They are generally unfaithful, and give us respectively the number and length of the sequential paths from one vertex of $S(G)$ to another, embedded in some path $P \in \Gamma(G)$. By the standard ring of additive quotients construction, both representations will define representations of the rings $\Delta(G)$ and $\Delta(S(G))$. Let $n = |V_S|$, we have a first representation in $\mathrm{M}_n(\mathbf{N})$, the semiring of $n \times n$ matrices with coefficients in the semiring of integers, via a bijective map $i \mapsto u_i$, $1 \leq i \leq n$, $u_i \in V_S$. Let $e_{ij} = (a_{kl}) \in \mathrm{M}_n(\mathbf{N})$ be the matrix whose entry a_{ij} is equal to 1, the others being null. We first define the morphism $\rho^1_S : \Gamma(S(G)) \to \mathrm{M}_n(\mathbf{N})$ by $\rho^1_S(x) = e_{ij}$ if $\partial^-(x) = u_i$ and $\partial^+(x) = u_j$, for x atomic, and we let $\rho^1 = \rho^1_S \circ \pi$. This morphism is well-defined and if $\rho^1(P) = (a_{kl})$, for all u_i, u_j in V_S, we have $a_{ij} = 0$ iff for all x, $y \in V$ with $\pi(x) = u_i$, $\pi(y) = v_j$, $m(P, x, y) = 0$. This implies that:

$$a_{ij} = \sum_{\substack{\pi(x)=u_i \\ \pi(y)=u_j}} m(P, x, y).$$

Now $b \circ \partial^-(P) = \{u_i\}$ iff the ith row of $\rho^1(P)$ is non-null, the other ones being null: if ρ^1 is extended to a ring map $\rho^1 : \Delta(G) \to \mathrm{M}_n(\mathbf{Z})$, the left ideal $e_{u_i}\Delta(G)$ is mapped into the left ideal of matrices whose jth row is null for $j \neq i$. Similarly, let $N = (\mathbf{N} \cup \{-\infty\}, \max, +, -\infty)$ be the semiring whose addition is the max operation and multiplication is the integer sum extended with $-\infty + x = -\infty$. The second representation $\rho^2_S : \Gamma(S(G)) \to \mathrm{M}_n(N)$, together with $\rho^2 = \rho^2_S \circ \pi$, is defined by:

- if e_{u_i} is idempotent, $\rho^2_S(e_{u_i}) = (a_{kl})$, with $a_{ii} = 0$ and $a_{kl} = -\infty$ if $k \neq i$ and $l \neq i$;
- if a is an atom, $\rho^2_S(a) = (a_{kl})$, with $a_{\partial^-(a)\partial^+(a)} = 1$ and $a_{kl} = -\infty$ if $k \neq \partial^-(a)$ and $l \neq \partial^+(a)$.

Let $\rho^2(P) = (a_{kl})$, $P \in \Gamma(G)$, then for all u_i, u_j in V_S, we have $a_{ij} = -\infty$ iff for all x, $y \in V$ with $\pi(x) = u_i$, $\pi(y) = v_j$, $m(P, x, y) = 0$. If $a_{ij} \neq -\infty$, a_{ij} is the length of the maximal sequential path in $Sub_s(P)$ embedded in P whose source lies over u_i and target over u_j. The length $|P| = \max(a_{kl})$ is then equal to $-\infty$ iff $P = 0$ and equal to 1 iff P is trivial non-null.

The rings $\Delta(G)$ are decomposed into direct sums of left ideals $\bigoplus_{e \in \mathcal{I}_G} e\Delta(G)$. Similarly in the homotopy rings $\Pi(G)$ and $\Sigma(G)$. For distinct atoms a and b in $\Delta(G)$, we also get a direct sum of left ideals $a\Delta(G) \oplus b\Delta(G)$. Further the ideal $a\Delta(G)$ is the disjoint union of the singleton $\{a\}$ and the direct sum $\bigoplus b\Delta(G)$, where b is an atom with $\partial^-(b) = \partial^+(a)$, $\overline{a}$ being such an atom (resp. right ideals). This implies that paths have a unique decomposition. More pragmatically, the underlying set of the algebra $\Gamma(G)$ is in bijective correspondence with the set of expressions S inductively defined by:

- 0, a belong to S, if a is atomic;
- $P_1 + P_2$ belong to S, if both P_1 and P_2 belong to S;
- $P_1 P_2$ belong to S, if both P_1, P_2 belong to S and $P_1 \restriction^+ (b_S \circ \partial^-_S(P_2)) = P_1$, $P_2 \restriction^- (b_S \circ \partial^+_S(P_1)) = P_2$.

This defines a unique decomposition of paths in $\Gamma(G)$. Similarly we get unique decompositions for contexts. This allows us to freely apply the syntactic tools to paths, the syntactic surgery being well-defined by this decomposition property.

3.3 Deduction Semantics

We now define both affine and projective paths, given a 2-graph G. An E-path is either of the form α, α a source atom or its inverse, or of the form E_1CE_2 for E-paths E_1 and E_2 and a C-path C, or of the form $a^{-1}Eb$ for an E-path E and atoms a, b with $l(a) = l(b)$. A C-path is either an idempotent, $aC_1c^{-1} + bC_2d^{-1}$ for C-paths C_1, C_2 and atoms a, b, c, d with $l(a) = l(c) = 0$, $l(b) = l(d) = 1$, for an E-path E and C-paths C_1,C_2, or an E-path. A P-path is of the form C, CE or EC for an E-path E and a C-path C.

An R-path is either an idempotent or of the form $aP_1R_1 + bP_2R_2$ where P_1, P_2 are P-paths, R_1, R_2 are R-paths, and a, b are atoms with $l(a) = 0$, $l(b) = 1$. The right non-strict affine paths are of the form ER, R a right strict affine path and E an E-path.

A right affine context is defined like a non-strict right affine path, where the clause generating the idempotents is replaced by a clause generating the trivial context. Such right contexts are *complete* in the sense that their holes single out all the targets of the denoted path. Left affine paths are the inverses of right affine paths, both strict and non-strict. Simple affine paths are defined by the condition that the P-paths that occur in the sum clause defining affine paths are both equal to an idempotent. By the unique decomposition property, any affine path possesses a unique associated simple affine path. The morphism f_G is injective on this class of connected paths.

The connected E-paths include *equality* paths of deduction semantics, the connected C-paths the *connection* paths of binary rules semantics. The P-paths range over *projective* paths, including both equality and connection paths. The R-paths (resp. L) are *right (*resp. *left) strict affine* paths, and include the right (resp. left) components of deductions semantical triple below. Under the involution, the set of projective paths is stable, while the set of L-paths and the set of R-paths are interchanged.

The decompositions of paths being unique, the definitions below are non-ambiguous. On strict affine paths, the idempotents are self-complementary; the left path $L_1P_1a^{-1} + L_2P_2b^{-1}$ and the right path $cQ_1R_1 + dQ_2R_2$ are complementary when both pairs (L_i, R_i), $i = 1, 2$, are, and $l(a) = l(c)$, $l(b) = l(d)$, a, b, c, d atoms. The *connection* transforms two complementary affine paths of a 2-graph algebra $\Gamma(G)$, a right one and a left one, into a projective path:

- $1|1 = 1$;
- $(aP_1R_1 + bP_2R_2)|(L_1P_1'c^{-1} + L_2P_2'd^{-1}) = aP_1(R_1|L_1)P_1'c^{-1} + bP_2(R_2|L_2)P_2'd^{-1}$, if $aP_1R_1 + bP_2R_2$ and $L_1P_1'c^{-1} + L_2P_2'd^{-1}$ are complementary right and left affine paths, with $l(a) = l(c)$, $l(b) = l(d)$.

Let us define the paths $A/_lO$, $A\backslash_lO$ of A (resp. $A/_rO$ and $A\backslash_rO$), given an occurrence O, which, in an equational setting, belongs to $\mathcal{O}(M)$, M the term described by the left (resp. right) connected strict affine path A. The path $A/_lO$ constructs the subterm M/O at occurrence O of the term M, whose one possible construction is A; while $A\backslash_lO$ denotes one construction, specified by A, of the sequential path in $S(\mathcal{E})$ from the source of M to the source of M/O.

- $L/_l\epsilon = L$;
- $(L_0P_0a_0^{-1} + L_1P_1a_1^{-1})/_l iO = L_j/_lO$, if $l(a_j) = i$;
- $L\backslash_l\epsilon = 1$;
- $(L_0P_0a_0^{-1} + L_1P_1a_1^{-1})\backslash_l i0 = (L_j\backslash_lO)P_ja_j^{-1}$, if $l(a_j) = i$;

And similarly for the definition of $/_r$ and $\backslash_r$.

Let $\mathcal{E}$ be some set of equations. Deductions are represented in the path algebra $\Gamma(\mathcal{E})$ by a semantic map $\Omega : \mathrm{Th}(\mathcal{E}) \to \Gamma(\mathcal{E})$. The right affine path $r(M) \in \Gamma(\mathcal{E})$ defined for occurrences of M in $\mathcal{E}$ is a source path from $\psi_M^{\mathcal{E}}(s)$, s the source of $T(M)$, to the vertices $\psi^{\mathcal{E}}(t)$, t a target $T(M)$. The vertex $\psi_M^{\mathcal{E}}(s)$ is noted $V_{\mathcal{E}}(M)$. The path $r(M)$ is defined by $r(x) = e_u$, if x is a variable, $u = V_{\mathcal{E}}(x)$; $r(f(M,N)) = ar(M) + br(N)$, if a (resp. b) is the atom associated to the edge from $V_{\mathcal{E}}(f(M,N))$ to $V_{\mathcal{E}}(M)$ (resp. $V_{\mathcal{E}}(N)$). We define $l(M) = r(M)^{-1}$. Similarly for contexts: we get $l(C[\_]) = D[\_]$, with $l(C[M]) = D[l(M)]$.

DEDUCTION	**SEMANTICS**
$\mathcal{D}_x$	(e_v, e_v, e_v), $v = V_{\mathcal{E}}(x)$,
$M = N$	$(l(M), \alpha, r(N))$
$S\,\dfrac{M = N}{N = M}$	$\dfrac{(L, E, R)}{(R^{-1}, E^{-1}, L^{-1})}$
$T\,\dfrac{M = N \quad N = O}{M = O}$	$\dfrac{(L_1, E_1, R_1) \quad (L_2, E_2, R_2)}{(L_1, E_1(R_1 \mid L_2)E_2, R_2)}$
$IL\,\dfrac{M = N \quad C[N] = O}{C[M] = O}$	$\dfrac{(L_1, E_1, R_1) \quad (L_2, E_2, R_2)}{(D_2[L_1E_1(R_1 \mid (L_2/_lO_C))], E_2, R_2)}$
$IR\,\dfrac{M = C[N] \quad N = O}{M = C[O]}$	$\dfrac{(L_1, E_1, R_1) \quad (L_2, E_2, R_2)}{(L_1, E_1, D_1[((R_1/_rO_C) \mid L_2)E_2R_2])}$
$E\,\dfrac{C[M] = D[N]}{M = N}$	$\dfrac{(L, E, R)}{(L/_lO, (L\backslash_lO)E(R\backslash_rO), R/_rO)}$

We associate a triple of paths to a deduction, as inductively specified in the above table. Equational edges define atoms in $\Gamma(\mathcal{E})$ denoted by lower-case greek letters: α is the source atom associated to $\rho_{\mathcal{E}}(M = N)$. In the semantics of introductions, the contexts are defined by the identities $D_2[L_2/_l O_C] = L_2$ and $D_1[R_1/_r O_C] = R_1$. In the last row, $O = O_C = O_D$ is the hole occurrence of the elimination proper contexts. The path L is the *left component* of $\Omega(\mathcal{D})$, R its *right component* and E its *equality component*. The semantics of $\mathcal{D}$ is the strongly connected path $\Omega(\mathcal{D}) = LER \in \Gamma(\mathcal{E})$. It is well-defined, i.e. single-valued by unique decomposition, and non-zero as $\partial^{\pm}(\Omega(\mathcal{D})) \neq 0$. The path L is a left affine target path from u_i to u, E is a projective 1-1 path from u to v, R is a right affine source path from v to v_j, where the u_i's (resp. v_j) are the targets of the construction specified by $\mathcal{D}$ of M (resp. N) in $T(\mathcal{E})$ and u (resp. v) to a root of M (resp. N). The vertices u_i and v_j are independant of $\mathcal{D}$, but not u and v. For both affine and projective paths in deduction semantics, the atoms in the sum clauses generating the paths are complementary: with the notations of the definition of a connection, in the sum clause, (a, b) are complementary as well as (c, d). The distinction elimination/introduction is semantically reflected: introductions built up new objects, while an elimination erases part of the information present in its premiss. Besides symmetry, the equality components of the premisses are subexpressions of the equality component of the conclusion of an inference rule.

4 Proof Theory

4.1 Normalization and Confluence

The path semantics defines a congruence on deductions: two deductions are equal iff their semantics are equal. The question of a purely syntactic definition of this equivalence introduces a finite set of equations. They are presented in natural deduction style. A formal translation in a first-order setting, with a formalization of context and occurrence calculus, is left to the reader. We distinguish five groups of equations Sym, Ide, Seq, Par, *Cut* and *Can*. They define a set of equations RE. Contexts are assumed to be non-trivial.

$$
Sym \quad \begin{cases}
S\,\dfrac{S\,\dfrac{M=N}{N=M}}{M=N} & \Longrightarrow & M=N & (1) \\[2ex]
S\,\dfrac{T\,\dfrac{M=N \quad N=O}{M=O}}{O=M} & \Longrightarrow & T\,\dfrac{S\,\dfrac{N=O}{O=N} \quad S\,\dfrac{M=N}{N=M}}{O=M} & (2) \\[2ex]
S\,\dfrac{IL\,\dfrac{M=N \quad C[N]=O}{C[M]=O}}{O=C[M]} & \Longrightarrow & IL\,\dfrac{S\,\dfrac{C[N]=O}{O=C[N]} \quad S\,\dfrac{M=N}{N=M}}{O=C[M]} & (3) \\[2ex]
S\,\dfrac{IR\,\dfrac{M=C[N] \quad N=O}{M=C[O]}}{C[O]=M} & \Longrightarrow & IR\,\dfrac{S\,\dfrac{N=O}{O=N} \quad S\,\dfrac{M=C[N]}{C[N]=M}}{C[O]=M} & (4) \\[2ex]
S\,\dfrac{E\,\dfrac{C[M]=D[N]}{M=N}}{N=M} & \Longrightarrow & E\,\dfrac{S\,\dfrac{C[M]=D[N]}{D[N]=C[M]}}{N=M} & (5)
\end{cases}
$$

$$
\mathit{Ide}\left\{
\begin{array}{lcll}
\mathit{IL}\dfrac{M=N \qquad \mathit{IR}\dfrac{C[N]=D[O] \quad O=P}{C[N]=D[P]}}{C[M]=D[P]} & \Longrightarrow & \mathit{IR}\dfrac{\mathit{IL}\dfrac{M=N \quad C[N]=D[O]}{C[M]=D[O]} \qquad O=P}{C[M]=D[P]} & (6) \\[4ex]
T\dfrac{T\dfrac{M=N \quad N=O}{M=O} \qquad O=P}{M=P} & \Longrightarrow & T\dfrac{M=N \qquad T\dfrac{N=O \quad O=P}{N=P}}{M=P} & (7) \\[4ex]
T\dfrac{\mathit{IR}\dfrac{M=C[N] \quad N=O}{M=C[O]} \qquad C[O]=P}{M=P} & \Longrightarrow & T\dfrac{M=C[N] \qquad \mathit{IL}\dfrac{N=O \quad C[O]=P}{C[N]=P}}{M=P} & (8) \\[4ex]
T\dfrac{\mathit{IL}\dfrac{M=N \quad C[N]=O}{C[M]=O} \qquad O=P}{C[M]=P} & \Longrightarrow & \mathit{IL}\dfrac{M=N \qquad T\dfrac{C[N]=O \quad O=P}{C[N]=P}}{C[M]=P} & (9) \\[4ex]
T\dfrac{M=N \qquad \mathit{IR}\dfrac{N=C[O] \quad O=P}{N=C[P]}}{M=C[P]} & \Longrightarrow & \mathit{IR}\dfrac{T\dfrac{M=N \quad N=C[O]}{M=C[O]} \qquad O=P}{M=C[P]} & (10)
\end{array}
\right.
$$

$$
\mathit{Seq}\left\{
\begin{array}{lcll}
\mathit{IL}\dfrac{\mathit{IL}\dfrac{M=N \quad C[N]=O}{C[M]=O} \qquad D[O]=P}{DC[M]=P} & \Longrightarrow & \mathit{IL}\dfrac{M=N \qquad \mathit{IL}\dfrac{C[N]=O \quad D[O]=P}{DC[N]=P}}{DC[M]=P} & (11) \\[4ex]
\mathit{IR}\dfrac{M=C[N] \qquad \mathit{IR}\dfrac{N=D[O] \quad O=P}{N=D[P]}}{M=CD[P]} & \Longrightarrow & \mathit{IR}\dfrac{\mathit{IR}\dfrac{M=C[N] \quad N=D[O]}{M=CD[O]} \qquad O=P}{M=CD[P]} & (12) \\[4ex]
\mathit{IL}\dfrac{\mathit{IR}\dfrac{M=C[N] \quad N=O}{M=C[O]} \qquad DC[O]=P}{D[M]=P} & \Longrightarrow & \mathit{IL}\dfrac{M=C[N] \qquad \mathit{IL}\dfrac{N=O \quad DC[O]=P}{DC[N]=P}}{D[M]=P} & (13) \\[4ex]
\mathit{IR}\dfrac{M=CD[N] \qquad \mathit{IL}\dfrac{N=O \quad D[O]=P}{D[N]=P}}{M=C[P]} & \Longrightarrow & \mathit{IR}\dfrac{\mathit{IR}\dfrac{M=CD[N] \quad N=O}{M=CD[O]} \qquad D[O]=P}{M=C[P]} & (14) \\[4ex]
\mathit{IL}\dfrac{T\dfrac{M=N \quad N=O}{M=O} \qquad C[O]=P}{C[M]=P} & \Longrightarrow & \mathit{IL}\dfrac{M=N \qquad \mathit{IL}\dfrac{N=O \quad C[O]=P}{C[N]=P}}{C[M]=P} & (15) \\[4ex]
\mathit{IR}\dfrac{M=C[N] \qquad T\dfrac{N=O \quad O=P}{N=P}}{M=C[P]} & \Longrightarrow & \mathit{IR}\dfrac{\mathit{IR}\dfrac{M=C[N] \quad N=O}{M=C[O]} \qquad O=P}{M=C[P]} & (16)
\end{array}
\right.
$$

$$\mathit{Par}\begin{cases}
\mathit{IR}\,\dfrac{\mathit{IR}\,\dfrac{M=C[N_1,N_2]\quad N_2=O_2}{M=C[N_1,O_2]}\quad N_1=O_1}{M=C[O_1,O_2]} \Longrightarrow \mathit{IR}\,\dfrac{\mathit{IR}\,\dfrac{M=C[N_1,N_2]\quad N_1=O_1}{M=C[O_1,N_2]}\quad N_2=O_2}{M=C[O_1,O_2]} & (17)\\[2ex]
\mathit{IL}\,\dfrac{M_2=N_2\quad \mathit{IL}\,\dfrac{M_1=N_1\quad C[N_1,N_2]=O}{C[M_1,N_2]=O}}{C[M_1,M_2]=O} \Longrightarrow \mathit{IL}\,\dfrac{M_1=N_1\quad \mathit{IL}\,\dfrac{M_2=N_2\quad C[N_1,N_2]=O}{C[N_1,M_2]=O}}{C[M_1,M_2]=O} & (18)
\end{cases}$$

$$\mathit{Cut}\begin{cases}
E\,\dfrac{\mathit{IL}\,\dfrac{C[M]=N\quad D[N]=EF[O]}{DC[M]=EF[O]}}{M=O} \Longrightarrow E\,\dfrac{T\,\dfrac{C[M]=N\quad E\,\dfrac{D[N]=EF[O]}{N=F[O]}}{C[M]=F[O]}}{M=O} & (19)\\[2ex]
E\,\dfrac{\mathit{IR}\,\dfrac{CD[M]=E[N]\quad N=F[O]}{CD[M]=EF[O]}}{M=O} \Longrightarrow E\,\dfrac{T\,\dfrac{E\,\dfrac{CD[M]=E[N]}{D[M]=N}\quad N=F[O]}{D[M]=F[O]}}{M=O} & (20)\\[2ex]
E\,\dfrac{\mathit{IL}\,\dfrac{M=N\quad C[N]=D[O]}{C[M]=D[O]}}{M=O} \Longrightarrow T\,\dfrac{M=N\quad E\,\dfrac{C[N]=D[O]}{N=O}}{M=O} & (21)\\[2ex]
E\,\dfrac{\mathit{IR}\,\dfrac{C[M]=D[N]\quad N=O}{C[M]=D[O]}}{M=O} \Longrightarrow T\,\dfrac{E\,\dfrac{C[M]=D[N]}{M=N}\quad N=O}{M=O} & (22)\\[2ex]
E\,\dfrac{\mathit{IL}\,\dfrac{M=N\quad CD[N]=E[O]}{CD[M]=E[O]}}{D[M]=O} \Longrightarrow \mathit{IL}\,\dfrac{M=N\quad E\,\dfrac{CD[N]=E[O]}{D[N]=O}}{D[M]=O} & (23)\\[2ex]
E\,\dfrac{\mathit{IR}\,\dfrac{C[M]=DE[N]\quad N=O}{C[M]=DE[O]}}{M=E[O]} \Longrightarrow \mathit{IR}\,\dfrac{E\,\dfrac{C[M]=DE[N]}{M=E[N]}\quad N=O}{M=E[O]} & (24)
\end{cases}$$

$$\mathit{Can}\begin{cases}
E\,\dfrac{\mathit{IL}\,\dfrac{M=N\quad C[N,O]=D[P]}{C[M,O]=D[P]}}{O=P} \Longrightarrow E\,\dfrac{C[N,O]=D[P]}{O=P} & (25)\\[2ex]
E\,\dfrac{\mathit{IR}\,\dfrac{C[M]=D[N,O]\quad N=P}{C[M]=D[P,O]}}{M=O} \Longrightarrow E\,\dfrac{C[M]=D[N,O]}{M=O} & (26)\\[2ex]
E\,\dfrac{\mathit{IL}\,\dfrac{M=N\quad C[O,N]=D[P]}{C[O,M]=D[P]}}{O=P} \Longrightarrow E\,\dfrac{C[O,N]=D[P]}{O=P} & (27)\\[2ex]
E\,\dfrac{\mathit{IR}\,\dfrac{C[M]=D[O,N]\quad N=P}{C[M]=D[O,P]}}{M=O} \Longrightarrow E\,\dfrac{C[M]=D[O,N]}{M=O} & (28)
\end{cases}$$

Theorem 4.1 [35] **Normalization in** LE: *The set of equations* RE *is well-founded and confluent. Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be two deductions,* $\mathcal{D} =_{RE} \mathcal{D}'$ *implies* $\Omega(\mathcal{D}) = \Omega(\mathcal{D}')$.

Proof. Local confluence follows from a mechanical critical pair checking according to the Newman and Knuth-Bendix lemmas. The relation generated by RE is well-founded: the number of binary inferences is constant or decreases under replacement. Hence in any derivation sequence, finite or infinite, the *Can* group is eventually no longer used. Hence we assume that derivation sequences are can-free. Similarly, by considering the distance of the symmetry rule instances to the sources, a *Sym* rule is eliminated from reduction sequences. Let the size of a deduction be the number of its binary rules instances and a subdeduction be secondary iff its conclusion is non-principal premiss of an introduction. Then the sum of the sizes of secondary subdeductions is either constant or decreases. Hence, in any reduction sequence the rules from the *Seq* and *Cut* groups, except rules (23), (24), are eventually no longer used. These two rules are eliminated by considering the sum of the sizes of subdeductions whose conclusion is premiss of an elimination. Next, counting the number of right introductions eliminates rule (8). The sum of the sizes of subdeductions whose conclusion is principal premiss of a right introduction decreases or is constant. This eventually eliminates rules (6) and (10). Similarly with the left introduction, which eliminates rule (9). There remains rule (7) and the *Par* group, which are clearly normalizing. Hence any derivation sequence is finite. The second part of the theorem follows from the equality of the semantics of both sides of rules, given below in §4.2 for the principal groups.□

By convention, two successive eliminations are merged into a single one: this can be done formally by adding an equation to RE, without destroying its normalization and confluence properties. The system differs from the one presented in [35] by the group *Seq*, whose equations are here inverted. Empirical evidence shows that only these two systems are interesting. Computations exhibit other finite systems, which however are non-symmetric in the processing of left and right introductions and only differ from the above systems by the equations orientation. Further, their *Cut* and *Can* groups are the same and we did not investigate their normalizing properties. The normal forms possess the well-known property found in natural deduction: introductions follow eliminations [18]. Let $\mathcal{N}$ be the deduction displayed at the top of next page, where multiple premisses are allowed, $n \geq 0$ and $m_n \geq 0$. Notice that $\mathcal{N}$ does not end by an introduction, but can be concluded by either an elimination or a transitivity rule. A *block* in a deduction is a principal branch of introductions, either all left or all right, such that the two occurrences of successive introductions are prefix one of the other. A normal form $\mathcal{D}$, according to the

$$
T\,\dfrac{\begin{array}{c}\mathcal{F}^n_1\\ F_{n-1}[Q^{n-1}_{m_{n-1}}] = Q^n_1\end{array} \;\cdots\; \begin{array}{c}\mathcal{F}^n_{m_n}\\ Q^n_{m_n-1} = Q^n_{m_n}\end{array} \quad E\,\dfrac{F_n[Q^n_{m_n}] = E_n \cdots E_1 C[(N_i)]}{Q^n_{m_n} = E_{n-1} \cdots E_1 C[(N_i)]}}{F_{n-1}[Q^{n-1}_{m_{n-1}}] = E_{n-1} \cdots E_1 C[(N_i)]}
$$

$$
\vdots
$$

$$
T\,\dfrac{\begin{array}{c}\mathcal{F}^1_1\\ D[(M_j)] = Q^1_1\end{array} \;\cdots\; \begin{array}{c}\mathcal{F}^1_{m_1}\\ Q^1_{m_1-1} = Q^1_{m_1}\end{array} \qquad E\,\dfrac{F_1[Q^1_{m_1}] = E_1 C[(N_i)]}{Q^1_{m_1} = C[(N_i)]}}{D[(M_j)] = C[(N_i)]}
$$

system RE of [35], is described by the deduction (with some conditions on $\mathcal{D}_i$ and $\mathcal{E}_j$):

$$
IR\,\dfrac{IL\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ P_1 = M_1\end{array} \;\cdots\; \begin{array}{c}\mathcal{D}_l\\ P_l = M_l\end{array} \qquad \begin{array}{c}\mathcal{N}\\ D[M_1, \ldots, M_l] = C[(N_i)]\end{array}}{D[P_1, \ldots, P_l] = C[N_1, \ldots, N_k]} \quad \begin{array}{c}\mathcal{E}_1\\ N_1 = O_1\end{array} \;\cdots\; \begin{array}{c}\mathcal{E}_k\\ N_k = O_k\end{array}}{D[P_1, \ldots, P_l] = C[O_1, \ldots, O_k]}
$$

According to the present system RE, a normal form is still displayed like $\mathcal{D}$ above, where the deductions $\mathcal{D}_i$ (resp. $\mathcal{E}_j$) can be nested. The subgraph equal to the branch from the vertex of $D[M_1, \ldots, M_l] = C[(N)]$ to the vertex of $D[P_1, \ldots, P_l] = C[(N)]$, together with the full subgraphs $\mathcal{D}_i$, is replaced by a similar subgraph, where the sequence $\mathcal{D}_1, \ldots, \mathcal{D}_l$ is now the concatenation of a sequence of blocks (resp. $\mathcal{E}_j$). The subdeductions $\mathcal{D}_i$ and $\mathcal{E}_j$, $i = 1, \ldots, l$, $j = 1, \ldots, k$, and $\mathcal{F}^p_q$, $p = 1, \ldots, n$, $q = 1, \ldots, m_p$ are in normal form and their last inference is not binary. The subdeduction $\mathcal{N}$ is the *principal subdeduction* of $\mathcal{D}$ while the $\mathcal{D}_i$'s and $\mathcal{E}_j$'s are its *auxiliary* deductions. This system minimizes the size of auxiliary deductions (and increases their number) through rewriting, while the previous system of [35] maximized this size. The leftmost (resp. rightmost) principal hypothesis of $\mathcal{D}$ is the leftmost (resp. rightmost) hypothesis of the deduction $\mathcal{N}$. The present choice of orientation for the *Ide* group implies that no introduction occurs in the leftmost principal branch of a normal form, and only left introductions appear in its rightmost principal branch.

In the logic LE, cuts are removed by the *Cut* equations. Contracting a cut may create a new one, whose distance to the sources is strictly smaller than the same distance for the contracted cut. In the *Seq* group, any left or right-hand side performs two substitutions in the same member of the conclusion. Equivalently, its equations possess a single principal branch, whose hypothesis is either the leftmost or the rightmost premiss of the rule. This group increases the length of principal branches. Due to equations (9) and (10), a cut or a cancellative redex can be potentially present in some deduction while not being explicit. The elimination of such a *phantom* redex is well-defined in any deduction of an RE-class where the redex is not explicit: $\mathcal{D}_1 \Rightarrow \mathcal{D}_2$ implies that any elimination instance in $\mathcal{D}_2$ unambiguously defines a unique elimination instance in $\mathcal{D}_1$. Also a deduction is cut-free (resp. can-free, sym-free, etc) if itself and all its derived deductions are Cut (resp. Can, Sym, etc) redexes free. Cuts excepted, the equations may modify the shape and length of principal branches. However, they preserve their number, their relative position and the equations at the two ends of the branches. The cut equations (19) to (22) create a new principal branch. Also, the notion of principal branch and hypothesis is a semantical one, syntactically well-defined for cut-free deductions only.

The set of equality subcomponents of a deduction $\mathcal{D}$ can be inductively defined, and includes all equality components of its subdeductions as well as the expressions $E_1(R_1|(L_2/_l O_C))$ and $((R_1/_r O_C)|L_2)E_2$ in the definition of introductions semantics. The set of *equality subcomponents* of a deduction *path* $\Omega(\mathcal{D})$ is the set of its maximal subexpressions that decompose into a product of equality subcomponents of the deduction itself. A deduction is sequential iff no equality subcomponent of its semantics contains a sum of two non-trivial elements. Hence, sequentiality is a property of RE-classes. The proper terms of a deduction are not invariant under replacement: the equations (8), (13) and (14) modify them by a context shift: proper terms N and $C[O]$ of a left-hand side become $C[N]$ and O in the right-hand side. From now on, all deductions are assumed to be sym and can-free. This last assumption has the following consequence. From the structure of normal forms and by induction on RE-derivations, each proper context of the premiss of an elimination in a can-free deduction belongs to a single hypothesis.

4.2 Geometric Realization of the Path Algebras

Let us briefly discuss the above definitions. We give the semantics of equations in the principal groups of RE: Ide, Seq and Cut, which thus completes the proof of theorem 4.1. The semantics of a term is denoted by the same letter, in bold face when projective. For example, related to a binary inference rule, $\mathbf{N}$ denotes the path $R|L$ where R (resp. L) is the semantics of the left (resp. right) occurrence of its proper term N, which is a right (resp. left) affine path, and $\mathbf{C}$ unambiguously denotes the path P, $P\backslash_l O_C$ or $P\backslash_r O_C$ if P is the semantics of a context $C[\_]$ in a left or right member of some equation. The equality component of subdeductions are denoted by greek letters.

$$C[M.\alpha.\mathrm{N}].(\beta.D[\mathbf{O}.\gamma.P]) = (C[M.\alpha.\mathrm{N}].\beta).D[\mathbf{O}.\gamma.P] \tag{6}$$

$$(M.\alpha.\mathrm{N}.\beta).\mathbf{O}.\gamma.P = M.\alpha.\mathrm{N}.(\beta.\mathbf{O}.\gamma.P) \tag{7}$$

$$M.\alpha.\mathbf{C}[(\mathrm{N}.\beta).\mathbf{O}].\gamma.P = M.\alpha.\mathbf{C}[\mathrm{N}.(\beta.\mathbf{O})].\gamma.P \tag{8}$$

$$(C[M.\alpha.\mathrm{N}].\beta).\mathbf{O}.\gamma.P = C[M.\alpha.\mathrm{N}].(\beta.\mathbf{O}.\gamma.P) \tag{9}$$

$$M.\alpha.\mathrm{N}.(\beta.C[\mathbf{O}.\gamma.P]) = (M.\alpha.\mathrm{N}.\beta).C[\mathbf{O}.\gamma.P] \tag{10}$$

$$D[(C[M.\alpha.\mathrm{N}].\beta).\mathbf{O}].\gamma.P = D[C[M.\alpha.\mathrm{N}].(\beta.\mathbf{O})].\gamma.P \tag{11}$$

$$M.\alpha.C[\mathrm{N}.(\beta.D[\mathbf{O}.\gamma.P])] = M.\alpha.C[(\mathrm{N}.\beta).D[\mathbf{O}.\gamma.P]] \tag{12}$$

$$D[M.\alpha.\mathbf{C}[(\mathbf{N}.\beta).\mathbf{O}]].\gamma.P = D[M.\alpha.\mathbf{C}[\mathbf{N}.(\beta.\mathbf{O})]].\gamma.P \quad (13)$$

$$M.\alpha.C[\mathbf{D}[\mathbf{N}.(\beta.\mathbf{O})].\gamma.P] = M.\alpha.C[\mathbf{D}[(\mathbf{N}.\beta).\mathbf{O}].\gamma.P] \quad (14)$$

$$C[(M.\alpha.\mathbf{N}.\beta).\mathbf{O}].\gamma.P = C[M.\alpha.\mathbf{N}.(\beta.\mathbf{O})].\gamma.P \quad (15)$$

$$M.\alpha.C[\mathbf{N}.(\beta.\mathbf{O}.\gamma.P)] = M.\alpha.C[(\mathbf{N}.\beta).\mathbf{O}.\gamma.P] \quad (16)$$

$$M.\mathbf{C}.(\alpha.(\mathbf{N}.(\mathbf{D}.\beta.\mathbf{E}))).\mathbf{F}.O = M.\mathbf{C}.((\alpha.\mathbf{N}).(\mathbf{D}.\beta.\mathbf{E})).\mathbf{F}.O \quad (19)$$

$$M.\mathbf{D}.(((\mathbf{C}.\alpha.\mathbf{E}).\mathbf{N}).\beta).\mathbf{F}.O = M.\mathbf{D}.((\mathbf{C}.\alpha.\mathbf{E}).(\mathbf{N}.\beta)).\mathbf{F}.O \quad (20)$$

$$M.(\alpha.(\mathbf{N}.(\mathbf{C}.\beta.\mathbf{D}))).O = M.((\alpha.\mathbf{N}).(\mathbf{C}.\beta.\mathbf{D})).O \quad (21)$$

$$M.(((\mathbf{C}.\alpha.\mathbf{D}).\mathbf{N}).\beta).O = M.((\mathbf{C}.\alpha.\mathbf{D}).(\mathbf{N}.\beta)).O \quad (22)$$

$$D'[M.((\alpha.\mathbf{N}).a^{-1}) + b^{-1}].\mathbf{C}.\beta.\mathbf{E}.O = D'[M.(\alpha.(\mathbf{N}.a^{-1})) + b^{-1}].\mathbf{C}.\beta.\mathbf{E}.O \quad (23)$$

$$M.\mathbf{C}.\alpha.\mathbf{D}.E'[a + (b.(\mathbf{N}.\beta)).O] = M.\mathbf{C}.\alpha.\mathbf{D}.E'[a + ((b.\mathbf{N}).\beta).O] \quad (24)$$

For the two equations (23) and (24), both contexts $D'[\_]$ and $E'[\_]$ are defined by the equalities $D'[M.((\alpha.\mathrm{N}).a^{-1}) + b^{-1}] = D[M.\alpha.N]$ and $E'[a + b.\mathrm{N}.\beta.O] = E[N.\beta.O]$. The *Sym* group express on deductions the algebraic laws of the inverse operation. Similarly, both groups *Ide* and *Seq* are related to the associativity law. Here is an informal explanation of this relation, restricted to affine and projective paths. We may classify the product into a left product, (l), of an affine path with a projective one, a right product, (r), of a projective path with an affine one and a purely projective product, (p). No affine-affine product appears in relation to associativity within semantics. In an instance of the associativity law $x.(y.z) = (x.y).z$, the proper element is the instance of y. This law occurs in the definition of semantics with a proper element which is either an equality component (*LER* product, introductions and elimination) or a connection path (transitivity). When the associativity law is so restricted, the equations of the *Ide* and *Seq* groups enumerate its various instances. In all equations from *Ide* and *Seq*, the proper element is β, the equality component of the second hypothesis. In the *Ide* group, the associativity instance includes the equational component of the semantics of the whole left or right-hand side, while it properly occurs in its left or right affine component in the *Seq* group. The combinations (p,l) and (r,p) for the product types in associativity are excluded and (r,l) does not appear in semantics. Hence, the proper element is necessarily projective. All cases are given in the following table.

Ide	*Seq*	*Cut*	Product Types
(6)			(l,r)
(7)	(15), (16)	(19)–(22)	(p,p) context-free
(8)	(13), (14)		(p,p) with context
(9)	(11)	(24)	(l,p)
(10)	(12)	(23)	(p,r)

When the proper element is a connection and besides the transitivity rule, we get cases of the form $a(N_1|N_2)E$ in left components and $E(N_1|N_2)a^{-1}$ in right ones, E an equality component and $a \in \mathcal{G}$, such as in equations (23) and (24). The *Cut* group is related to such instances of associativity. Similarly, the *Par* group expresses both the commutativity and the associativity (arbitrary arity operators) of the sum. Notice that the partition in the above table is unstable under associativity.

An alternative definition of the path and homotopy rings, close to definitions of algebraic topology [49, 55], where an element of the homotopy group is defined as a class of homotopy equivalent morphisms, stumbles over the existence of the sum and over non-connectedness. The essential point here is the existence of homotopy groups embedded in these rings. Now, a geometric definition of $\Gamma(G)$ would require the construction of deduction paths as morphisms from specific graphs that replace the standard chains of sequential paths. Such graphs will be useful in characterizing cuts by a discrete variational principle for deduction semantic paths.

A term M is *linear* iff its term graph $T(M)$ is a tree. The term M linearizes the term N iff there exists an onto equational morphism $\phi : T(M) \rightarrow T(N)$. Up to isomorphism, a term possesses a unique *linearization*, which is a linear term that linearizes it. These definitions also apply to equations. Given such an object O, term or equation, and one of its linearization O', we get a unique geometric substitution σ such that $\sigma(O') = O$. A deduction $\mathcal{D}$ from the set of equations $\mathcal{E}$ is linear iff every equation in $\mathcal{E}$ is associated to at most one source of $\mathcal{D}$. Finally, the path P in a path algebra Γ is linear iff

- either $P = 0$ or P is atomic;
- or $P = P_1 + P_2$ or $P = P_1 P_2$, both P_1 ,P_2 are linear and $Sub(P_1) \cap Sub(P_2)$ is a set of idempotents of Γ.

This notion is well-defined by the unique decomposition property of paths. A set of equations $\mathcal{E}'$ locally sequentializes the set of equations $\mathcal{E}$ iff $\mathcal{E}'$ is equal to $\mathcal{E}$ where all occurrences of some non-variable term having at least two distinct occurrences in $\mathcal{E}$ are replaced by some variable not belonging to $\mathcal{V}(\mathcal{E})$. A set of equations $\mathcal{E}$ is *sequential* iff there does not exist any set of equations that locally sequentializes it. Up to variable permutation, any set of equations defines a unique sequential set of equations, called its *sequentialization*.

Definition 4.1 *The set of equations $\mathcal{E}$ is quasi-linear iff equations in $\mathcal{E}$ are linear, and $T(\mathcal{E}')$ is a tree, where $\mathcal{E}'$ is the sequentialization of $\mathcal{E}$. It is linear if further variables in $\mathcal{V}(\mathcal{E})$ possess at most two occurrences in $\mathcal{E}$.*

Let now $\mathcal{D} \vdash^{\mathcal{E}}_{LE} M = N$ be a deduction. We construct a linear deduction $\mathcal{D}' \vdash^{\mathcal{E}'}_{LE} M' = N'$ and a substitution σ' such that $\sigma'(\mathcal{D}') = \mathcal{D}$, $dom(\sigma) \subseteq \mathcal{V}(\mathcal{D}')$, geometric on $\mathcal{W}(\mathcal{D}')$, and possibly relabelling. The deduction $\mathcal{D}'$ is the *linearization* of $\mathcal{D}$, unique up to isomorphism.

The set $\mathcal{A}(\mathcal{D}') = \mathcal{E}_{\mathcal{D}}$ is linear, and M' (resp. N') is a linearization of M (resp. N). The pair $(\mathcal{D}', \sigma')$ is universal: for any pair $(\mathcal{D}'', \sigma'')$ fulfilling these conditions, there exists σ such that $\sigma(\mathcal{D}') = \mathcal{D}''$, hence $\sigma' = \sigma'' \circ \sigma$. From a pair $(\mathcal{D}', \sigma')$, we get an equational morphism $\phi' : T(\mathcal{E}_{\mathcal{D}}) \to T(\mathcal{E})$, which is the parallel analogue of morphisms from standard chains. A context $C[\_]$ is *filiform* whenever the non-trivial occurrence O belongs to $\mathcal{O}(C[\_])$ iff either O or its complement $\overline{O}$ prefixes O_C. Up to isomorphism, any context $C[\_]$ defines a unique linear filiform context $D[\_]$ with $O_C = O_D$ called its linearization, and a substitution σ such that $\sigma(D[\_]) = C[\_]$, not necessarily geometric. The deduction $\mathcal{D}'$ is inductively defined on $\mathcal{D}$ as follows.

- If $\mathcal{D}$ is empty, so is $\mathcal{D}'$. If $\mathcal{D}$ reduces to a single hypothesis, $\mathcal{D}'$ is a linearization of this hypothesis, σ' and ϕ' are unambiguously defined.

- If $\mathcal{D}$ ends up with an elimination, let $\mathcal{C} \vdash C[M] = D[N]$, $O_C = O_D$, be the premiss of this elimination and $\mathcal{C}' \vdash C'[M'] = D'[N']$ be a linearization of $\mathcal{C}$. Then $C'[\_]$ belongs to some hypothesis, say $E[C'[P]] = R$. So does $D'[\_]$, say $S = F[D'[Q]]$ (up to the symmetry rule). The deduction $\mathcal{D}'$ is the deduction obtained by adding to $\mathcal{C}'$ the elimination of occurrence O_C and by replacing the hypotheses $E[C'[P]] = R$ and $S = F[D'[Q]]$ by $E[C''[P]] = R$ and $S = F[D''[Q]]$, where $C''[\_]$ and $D''[\_]$ are two linearizations of $C'[\_]$ and $D'[\_]$ respectively, such that $\mathcal{V}(C''[\_]) \cap \mathcal{V}(D''[\_]) = \emptyset$, and e.g. $\mathcal{V}(C''[\_]) \cap \mathcal{V}(\mathcal{C}') = \mathcal{V}(D''[\_]) \cap \mathcal{V}(\mathcal{C}') = \emptyset$. As above, both σ' and ϕ' are unambiguously defined from the corresponding data for $\mathcal{C}'$.

- If $\mathcal{D}$ ends up with a binary inference rule, e.g. a transitivity instance:

  $$T\,\frac{\overset{\mathcal{D}_1}{M = N} \qquad \overset{\mathcal{D}_1}{N = O}}{M = O}, \quad \text{we build the deduction } \mathcal{D}'\text{:} \quad T\,\frac{\overset{\sigma(\mathcal{D}'_1)}{\sigma(M'_1) = N'_2} \qquad \overset{\mathcal{D}'_2}{N'_2 = O'_2}}{\sigma(M'_1) = O'_2}$$

  where $\mathcal{D}'_1 \vdash M'_1 = N'_1$ is a linearization of $\mathcal{D}_1$, $\mathcal{D}'_2 \vdash M'_2 = N'_2$ one of $\mathcal{D}_2$, with $\mathcal{V}(\mathcal{D}'_1) \cap \mathcal{V}(\mathcal{D}'_2) = \emptyset$, and σ a permutation with minimal domain such that $\sigma(N'_1) = N'_2$, which exists because both $\mathcal{A}(\mathcal{D}'_1)$ and $\mathcal{A}(\mathcal{D}'_2)$ are linear, hence both N'_1 and N'_2 also. Once more both σ' and ϕ' are easily defined.

Similarly, let $G = (V, E, \partial^-, \partial^+)$ be a 2-graph and $P \in \Gamma(G)$ some path. The *linearization* of P is a linear path $Q \in \Gamma(T(P))$, for a graph $T(P)$, together with a morphism $\phi : T(P) \to G$ such that, given the corresponding data $(T(\pi_\sharp(P)), Q', \psi)$ for the path $\pi_\sharp(P)$ and the graph $S(G)$, there exists a map $\pi_P : T(P) \to T(\pi_\sharp(P))$ with $\pi \circ \phi = \psi \circ \pi_P$. In general, both graph morphisms ϕ and ψ do not define direct image morphisms. However they do provide inductively defined maps ϕ' and ψ' mapping connected paths to connected paths, which are not morphisms. We then have $\phi'(Q) = P$ and $\psi'(Q') = \pi_\sharp(P)$. Naturally, we also have $\pi_{P_\sharp}(Q) = Q'$. We specify the construction of the morphism

$\pi_P : T(P) \to T(\pi_\sharp(P))$, $P \in \Gamma(G)$ and of the paths Q and Q', belonging respectively to the $\pi_{P\sharp}$-related algebras $\Gamma(T(P))$ and $\Gamma(T(\pi_\sharp(P)))$:

- The graph $T(0)$ is the empty graph, $Q = 0$ and ϕ is the empty map.
- If P is idempotent, both $T(P)$ and $T(\pi_\sharp(P))$ are graphs having a single vertex and no edges, π_P being non ambiguous. The morphism ϕ maps this vertex v to the vertex associated to P in G, and ψ maps the unique vertex of $T(\pi_\sharp(P))$ to $\pi(\phi(v))$. The paths Q and Q' are respectively equal to the unique idempotent of $\Gamma(T(P))$ and of $\Gamma(T(\pi_\sharp(P)))$. Similarly for P an atom: the graphs $T(P)$ and $T(\pi_\sharp(P))$ possess two distinct vertices, and a unique edge that links them. Both Q and Q' are equal to the unique atom in the algebras $\Gamma(T(P))$ and $\Gamma(T(\pi_\sharp(P)))$.
- If $P = P_1 + P_2$, the graph $T(P)$ is the disjoint union of the two graphs $T(P_1)$ and $T(P_2)$. Similarly for the graph $T(\pi_\sharp(P_1 + P_2))$. The path Q is the sum $j_{1\sharp}(Q_1) + j_{2\sharp}(Q_2)$, Q_i associated to P_i, j_i the injection of $T(P_i)$ in $T(P)$, $i = 1, 2$. The graph of the morphism ϕ is isomorphic to the disjoint union of the graphs of the morphisms associated to P_i, $i = 1, 2$. The morphisms $j_{1\sharp}$ and $j_{1\sharp}$ exist as we have two graph injections of $T(\pi_\sharp(P_1))$ and $T(\pi_\sharp(P_2))$ into $T(\pi_\sharp(P_1 + P_2))$.
- If $P = P_1P_2$ is non-null, we assume that $P_1 \upharpoonright^+ (b_S \circ \partial_S^-(P_2)) = P_1$ and $P_2 \upharpoonright^- (b_S \circ \partial_S^+(P_1)) = P_2$. Hence $b_S \circ \partial_S^+(P_1) = b_S \circ \partial_S^-(P_2)$. Let $\phi_i : T(P_i) \to G$, $i = 1, 2$, be the maps inductively constructed. Let G_1 be the quotient graph of $T(P_1)$ obtained by identifying into v_j^1 all vertices in $\phi_1^{-1}(v_j)$, for every v_j in $b \circ \partial^+(P_1)$ (resp. G_2, $T(P_2)$, v_j^2, $b \circ \partial^-(P_2)$). The graph $T(P)$ is the quotient of the disjoint union of G_1 and G_2 under the identification of v_j^1 with v_j^2 when $v_j \in b \circ \partial^+(P_1) \cap b \circ \partial^-(P_2)$; $Q = (j_1'(Q_1))(j_2'(Q_2))$, Q_i associated to P_i, j_i the map $T(P_i) \to G_i \to T(P)$, $i = 1, 2$ ($j_{i\sharp}$ does not exist in general, cf. the above remark); $\phi : T(P) \to G$ is easily constructed. The constructions for $\pi_\sharp(P_1P_2)$ is similar.

This construction is well-defined and coherent with the linearization of a deduction $\mathcal{D}$ from equations in $\mathcal{E}$, in the sense that $T(\mathcal{E}_\mathcal{D})$ is a minimal 2-graph that contains (an isomorphic copy of) the graph $T(\Omega(\mathcal{D}))$ of the linearization of $\Omega(\mathcal{D})$, when the edge labels of $T(\mathcal{E})$ are pulled-back to $T(\Omega(\mathcal{D}))$. A path $P \in \Gamma(G)$ is trivial iff the edge set of the graph $T(P)$ is empty. Given a deduction $\mathcal{D}$, on the graph $T(\mathcal{E}_\mathcal{D})$, the "projective space", equal to the minimum full subgraph of $T(\mathcal{E}_\mathcal{D})$ that includes all edges associated to equational paths, grows with deductions, *even* through eliminations.

For the relation with the usual, non-constructive, equational logic EL [22, 48], a set $\mathcal{E}$ of equations is flat iff no fixed-point equation $x = C[x]$, $C[\_]$ non-trivial, is deducible in LE from $\mathcal{E}$. Now if $\mathcal{D} \vdash_{EL}^{\mathcal{E}} M = N$, by duplication of equations and variable permutation, we may assume that two distinct equations in $\mathcal{E}$ do not share variables except two equations that occur as premisses of some transitivity instance in $\mathcal{D}$, and that an equation from

$\mathcal{E}$ has at most one occurrence as an hypothesis of $\mathcal{D}$. The substitutions in $\mathcal{D}$ can be chosen so that the variables in their range are disjoint from the variables in their domain. Hence, the relation $\mathcal{D} \vdash_{EL}^{\mathcal{E}} M = N$ implies that there exist a flat set of equations $\mathcal{E}'$, the union of the substitutions in $\mathcal{D}$ in equation form, such that $\vdash_{LE}^{\mathcal{E}\cup\mathcal{E}'} M = N$ (see [35] for the transformation of deductions in EL into deductions in LE). The constructivism and atomicity of LE implies that there exists deductions in this formal system without corresponding deductions in EL. Another interpretation of the graph $S^-(\mathcal{E})$, equal to $S(\mathcal{E})$ with the equational vertices and edges removed, is quite useful: this graph is nothing but the "Cayley graph" of the free algebra on the empty set, with operators including both f and "variables" from $\mathcal{V}(\mathcal{E})$ as constants, and satisfying the relations $\mathcal{E}$. We mention the relation with functional equations in typed λ-calculi. Consider the set of equations $\mathcal{E}_2$, with associated deduction $\mathcal{D}_2$:

$$\mathcal{E}_2 \left\{ \begin{array}{rcl} f(x,y) & = & x \\ x & = & f(f(x,z),t) \end{array} \right. \qquad E\,\dfrac{T\,\dfrac{f(x,y) = x \quad x = f(f(x,z),t)}{f(x,y) = f(f(x,z),t)}}{x = f(x,z)}$$

Consider now some associated second-order equations:

1. $f(\Pi(X),y) = X(A)$, $X(B) = f(f(\Pi(X),z),t)$,

2. $f(\Pi(X),y) = X(A)$, $X(A) = f(f(\Pi(X),z),t)$,

3. $\lambda y.f(X(A),y) = X$, $X = \lambda t.f(f(\Pi(X),z),t)$,

4. $\lambda a.f(X(a),y) = X$, $X(A) = f(f(\Pi(X),z),t)$.

The reader can check that all sets, except the second one, solve the fixed-point deduction $\mathcal{D}_2$, via the resolution rule. Assume that in $\mathcal{E}_2$ the two occurrences of x connected in $\mathcal{D}_2$ are replaced by a non-variable term $f(u,v)$. Simple examples of associated second-order equations demonstrate the necessity of a non-distributive algebra in order to get a correct modelization of the operationality of the resolution rule. Consider the two second-order terms $\lambda x,y.f(U(X(x),y),V(x,Y(y)))$ and $\lambda z,t.f(U(z,Y(t)),V(X(z),t))$, associated to the two occurrences of $f(u,v)$ in the modified set $\mathcal{E}_2$, they are locally unifiable, but not globally. On the semantics, this means that the operational behaviour of $f^{-1}\alpha(bc^{-1}+de^{-1})\beta^{-1}g(a+a')$ is distinct from its distributed form $f^{-1}\alpha bc^{-1}\beta^{-1}g(a+a') + f^{-1}\alpha de^{-1}\beta^{-1}g(a+a')$. No distributive algebraic structure seems to reflect this fact. Further, we did not turn the path algebra $\Gamma(G)$ into an abelian group: the sign rules $(-x)y = x(-y) = -(xy)$ require distributivity. This is also true for ideals: an "ideal" in $\Gamma(G)$ does not define a congruence, once more distributivity is required. Finally, we did not attempt to formalize an algebra with several sums and products, by lack of any operationality requirement.

5 Particle Systems on 2-Graphs

In this section the graph G is a 2-graph. However, the particle game takes its full meaning when G is a pograph. For in such a case, there exists a natural distribution of particles on the graph $T(P)$ of a path $P \in \Gamma(G)$.

5.1 The Transition Rules

States are members of a set $St(G)$, with an onto map $p : St(G) \to \Gamma(G)$, and an injection $i : \Gamma(G) \hookrightarrow St(G)$. Usual definitions on paths such as connectedness, source and target, source atoms, projective and affine states, etc, are pulled back to states via p. We assume the existence of a denumerable set $\mathcal{P}$ of particles, which is a disjoint union $\mathcal{P}^+ \sqcup \mathcal{P}^-$ of a set of positive particles $\mathcal{P}^+$, noted ε^+, in bijective correspondence with a set of negative particles $\mathcal{P}^-$, noted ε^-. The particles are thereby associated by pairs. A particle system on G is a sequence of states in some fiber $p^{-1}(P)$, $P \in \Gamma(G)$. Two successive states are related by specific transitions. A state represents a configuration of particles on the vertices of $T(P)$. However, it is easier to define this configuration on the liearization of P. The negative particle move towards the sources of this path $Q \in \Gamma(T(P))$ (resp. positive and targets). We let:

$$St(G) = \{(P, f^-, f^+) \mid P \in \Gamma(G),\ f^- : V_{T(P)} \to \mathfrak{P}(\mathcal{P}^-),\ f^+ : V_{T(P)} \to \mathfrak{P}(\mathcal{P}^+)\}.$$

The set of particles of a state σ is noted $\mathcal{P}(\sigma) = \mathcal{P}^+(\sigma) \sqcup \mathcal{P}^-(\sigma)$, $\mathcal{P}^\pm(\sigma) = \bigcup_{v \in V_{T(P)}} f^\pm(v)$.

The particles from $\mathcal{P}(\sigma)$, σ a state, split up into two types: the projective and the affine ones. Projective particles come into two kinds: source particles and target particles, sources and targets being those of G. The affine ones are related to sources and targets of P. Each pair contains a positive particle and a negative one. On transition rules below, a type, affine or projective, is assigned to particles sets: all particles in projective rules are projective, all particles of affine rules are affine except those of $\varepsilon_2^\pm$ (resp. $\varepsilon_1^\pm$) on the affine-projective (resp. projective-affine) rule left-hand side, which become affine in right-hand sides. The map $i : \Gamma(G) \hookrightarrow St(G)$ is defined by $i(P) = (P, f^-, f^+)$, with $f^\pm(v) = \emptyset$. When G is a 2-pograph, besides i, the *initial states* define another section $\sigma : \Gamma(G) \to St(G)$ of the projection p. For a path $P \in \Gamma(G)$, let $Q \in \Gamma(T(P))$ be its linearization and $\phi : T(P) \to G$, we have $\sigma(P) = (P, f^-, f^+)$ where $f^\pm$ is specified as follows.

- $f^+(v)$ contains an affine particle ε^+, $f^-(v)$ contains the associated particle ε^-, if v is either a source or a target of the path Q.

- $f^+(v)$ contains a target projective particle ε^+, $f^-(v)$ contains the associated target projective particle ε^-, if v is an internal vertex of $T(P)$, mapped to a target of G by ϕ.

- $f^+(v)$ contains a source projective particle ε^+ if v is a vertex of $T(P)$ mapped to a successor of a source of G by ϕ, and $f^-(u)$ contains the associated source projective particle ε^- if u is mapped to the other successor of the same source, and further, in $T(P)$, u and v are two successors of a source of $T(P)$.

- There are no other particles associated to $\sigma(P)$ and we require that particles associated to distinct vertices of $T(P)$ are distinct.

We assume that such a section σ is given once for all for each 2-graph G. Similarly, the map $j : \Gamma(G) \to St(G)$ is defined by $j(P) = (P, f^-, f^+)$ with $f^-(u) = \{\varepsilon^- \mid \varepsilon^- \in \mathcal{P}^-(\sigma(P))\}$ for $u \in b \circ \partial^-(Q)$ (respectively $+$), $f^\pm(w) = \emptyset$ otherwise.

Let now G be any 2-graph. We first describe the projective transitions: affine rules become projective, e.g. under a "projective embedding" of graphs, cf. §5. If ε_1^+ and ε_2^+ denote two sets of particles, their union is noted $\varepsilon_{1,2}^+$ (resp. ε^-). The variables a, b, c, d range over atoms, P, P_1 and P_2 over states and $\varepsilon^\pm$, $\varepsilon_1^\pm$, $\varepsilon_2^\pm$, $\varepsilon_{1,2}^\pm$ over sets of particles, with the above convention.

Projective System P

Product Rule : $$a^{-1}\varepsilon^- P \varepsilon^+ b \;\Rightarrow\; \varepsilon^- a^{-1} P b \varepsilon^+, \quad \text{if} \quad l(a) = l(b);$$

Sum Rule : $$a\varepsilon_1^- P_1 \varepsilon_1^+ b^{-1} \;+\; c\varepsilon_2^- P_2 \varepsilon_2^+ d^{-1} \Rightarrow \varepsilon_{1,2}^-(aP_1b^{-1} \;+\; cP_2d^{-1})\varepsilon_{1,2}^+,$$ $$\text{if} \quad l(a) = l(b) = 0,\ l(c) = l(d) = 1;$$

Exchange Rule : $$\varepsilon_1^- P_1 \varepsilon_1^+ \varepsilon_2^- P_2 \varepsilon_2^+ \;\Rightarrow\; \varepsilon_{1,2}^- P_1 P_2 \varepsilon_{1,2}^+.$$

The meaning of these rules is: under the exchange rule $(P, f^-, f^+) \Rightarrow (P, g^-, g^+)$ is a transition iff $P = C[P_1P_2]$, and accordingly $Q = D[Q_1Q_2]$, Q the linearization of P, $f^-(v) = \varepsilon_1^-$ for $v \in b \circ \partial^-(Q_1)$, $f^-(w) = \varepsilon_2^-$ for $v \in b \circ \partial^-(Q_2)$, and $g^-(v) = \varepsilon_{1,2}^-$, $g^-(w) = \emptyset$, $g^-(u) = f^-(u)$ if $u \notin b \circ \partial^-(Q_1) \cup b \circ \partial^-(Q_2)$ (resp. $+$). And similarly for the other rules. By contrast with the projective case, affine rules are non-linear with respect to particles. All contexts in affine rules are assumed to exhibit all occurrences of the displayed particle set in the redex of the reducible expression. We further require that these particle sets are associated to sources (resp. targets) in contexts noted $L[\_, \ldots, \_]$ or $L_i[\_, \ldots, \_]$ (resp. $R[\_, \ldots, \_]$ or $R_i[\_, \ldots, \_]$), $i = 1, 2$. The variables a, b range over atoms; $L[\_, \ldots, \_]$, $L_i[\_, \ldots, \_]$, $R[\_, \ldots, \_]$ and $R_i[\_, \ldots, \_]$, $i = 1, 2$, over state contexts; P over states and the variables ε over particle products as in the projective case.

Affine System A

Left Affine Rule : $$L_1[\varepsilon_1^-, \ldots, \varepsilon_1^-]\varepsilon_1^+ a^{-1} \;+\; L_2[\varepsilon_2^-, \ldots, \varepsilon_2^-]\varepsilon_2^+ b^{-1}$$ $$\Rightarrow \quad (L_1[\varepsilon_{1,2}^-, \ldots, \varepsilon_{1,2}^-]a^{-1} \;+\; L_2[\varepsilon_{1,2}^-, \ldots, \varepsilon_{1,2}^-]b^{-1})\varepsilon_{1,2}^+, \quad \text{if } l(a) = 0,\ l(b) = 1;$$

Right Affine Rule : $\qquad a\varepsilon_1^- R_1[\varepsilon_1^+,\ldots,\varepsilon_1^+] \;+\; b\varepsilon_2^- R_2[\varepsilon_2^+,\ldots,\varepsilon_2^+]$

$$\Rightarrow \quad \varepsilon_{1,2}^-(aR_1[\varepsilon_{1,2}^+,\ldots,\varepsilon_{1,2}^+] \;+\; bR_2[\varepsilon_{1,2}^+,\ldots,\varepsilon_{1,2}^+]), \quad \text{if } l(a)=0,\ l(b)=1;$$

Affine – Projective Rule : $\qquad L[\varepsilon_1^-,\ldots,\varepsilon_1^-]\varepsilon_1^+\varepsilon_2^- P\varepsilon_2^+ \quad\Rightarrow\quad L[\varepsilon_{1,2}^-,\ldots,\varepsilon_{1,2}^-]P\varepsilon_{1,2}^+;$

Projective – Affine Rule : $\qquad \varepsilon_1^- P\varepsilon_1^+\varepsilon_2^- R[\varepsilon_2^+,\ldots,\varepsilon_2^+] \quad\Rightarrow\quad \varepsilon_{1,2}^- PR[\varepsilon_{1,2}^+,\ldots,\varepsilon_{1,2}^+];$

Terminal Rule : $L[\varepsilon_1^-,\ldots,\varepsilon_1^-]\varepsilon_1^+\varepsilon_2^- R[\varepsilon_2^+,\ldots,\varepsilon_2^+] \quad\Rightarrow\quad L[\varepsilon_{1,2}^-,\ldots,\varepsilon_{1,2}^-]R[\varepsilon_{1,2}^+,\ldots,\varepsilon_{1,2}^+].$

The transition system PA is the union of the two systems P and A. In this last system A, we distinguish the *proper affine* rules: the left, the right and terminal rules, from the *mixed* rules: the affine-projective and the projective-affine rules. The *effective* rules are the product, the sum and the proper affine rules. The *logical* rules are the product, the exchange and the mixed rules, the remaining ones are the *structural* rules: the sum, affine and terminal rules. The *proper* particles of a redex are those displayed by the corresponding left-hand side. We say that these particles are *moved* by the transition. The various states in a redex that are instances of a variable in its PA-rule are its *proper* states, with the exception of the mixed rules: the instance of the variable P in a redex is its *improper* state, and of the context its proper state. Similarly, the atoms occurring in effective rules redexes, besides the particles and proper states, are also called proper. The transition system PA is well-defined on $St(G)$. A transition $\sigma \Rightarrow \rho$ implies $p(\sigma) = p(\rho)$ as well as $\mathcal{P}(\sigma) = \mathcal{P}(\rho)$. Also we may refer to common atoms of distinct states on some pebbling system, thereby referring to atoms in the common linearization Q of the support $p(\sigma)$ of the states in the system.

The condition source-target on contexts merely prevents affine rules to be applied to projective particles. Consider $\mathcal{E}_3 = \{x = f(y,z),\ z = f(a,b)\}$, and the deduction $\mathcal{D}_3$:

$$IL\,\frac{x = f(y,z) \qquad z = f(a,b)}{x = f(y, f(a,b))}$$

One may accept the following pebbling, where the right affine rule is applied to the projective particle $\varepsilon_3^\pm$ (notice that both ε_1 and ε_3 denote products of two particles):

$$\begin{aligned}
\sigma(\Omega(\mathcal{D}_3)) &\Rightarrow \varepsilon_0^-\varepsilon_0^+\gamma^{-1}\varepsilon_1^-\varepsilon_1^+\delta(a\varepsilon_2^-\varepsilon_2^+ + d\varepsilon_3^-\alpha^{-1}\beta\varepsilon_3^+(b\varepsilon_4^-\varepsilon_4^+ + c\varepsilon_5^-\varepsilon_5^+))\\
&\Rightarrow \varepsilon_0^-\varepsilon_0^+\gamma^{-1}\varepsilon_1^-\varepsilon_1^+\delta\varepsilon_{2,3}^-(a\varepsilon_{2,3}^+ + d\alpha^{-1}\beta\varepsilon_{2,3}^+(b\varepsilon_4^-\varepsilon_4^+ + c\varepsilon_5^-\varepsilon_5^+))\\
&\Rightarrow \varepsilon_0^-\varepsilon_0^+\gamma^{-1}\varepsilon_1^-\varepsilon_1^+\delta\varepsilon_{2,3}^-(a\varepsilon_{2,3}^+ + d\alpha^{-1}\beta\varepsilon_{2,3}^+\varepsilon_{4,5}^-(b\varepsilon_{4,5}^+ + c\varepsilon_{4,5}^+))\\
&\Rightarrow \varepsilon_0^-\varepsilon_0^+\gamma^{-1}\varepsilon_1^-\varepsilon_1^+\delta\varepsilon_{2,3,4,5}^-(a\varepsilon_{2,3,4,5}^+ + d\alpha^{-1}\beta(b\varepsilon_{2,3,4,5}^+ + c\varepsilon_{2,3,4,5}^+))\\
&\Rightarrow \cdots \Rightarrow \varepsilon_{0,1,2,3,4,5}^-\gamma^{-1}\delta(a\varepsilon_{0,1,2,3,4,5}^+ + d\alpha^{-1}\beta(b\varepsilon_{0,1,2,3,4,5}^+ + c\varepsilon_{0,1,2,3,4,5}^+)).
\end{aligned}$$

The system PA can be extended to encompass such cases, by adding some mixed rules. Another possible extension of the system PA would increase the number of particles, e.g. by adding two pairs per occurrence of the product operator. The present choice minimizes

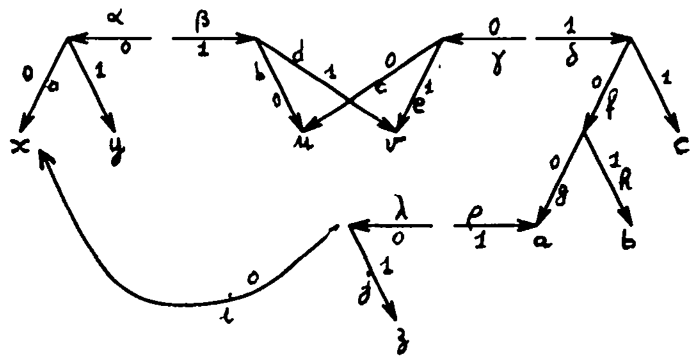

Figure 2: The Graph $T(\mathcal{E}_4)$

the number of particles. Consider $\mathcal{E}_4 = \{f(x,y) = f(u,v),\ f(u,v) = f(f(a,b),c),\ a = f(x,z)\}$, and the deduction $\mathcal{D}_4$:

$$IR\,\dfrac{E\,\dfrac{T\,\dfrac{f(x,y)=f(u,v)\quad f(u,v)=f(f(a,b),c)}{f(x,y)=f(f(a,b),c)}}{x=f(a,b)}\qquad a=f(x,z)}{x=f(f(x,z),b)}$$

From Fig. 2, we have $\Omega(\mathcal{D}_4) = a^{-1}\alpha^{-1}\beta(bc^{-1}+de^{-1})\gamma^{-1}\delta f(g\rho^{-1}\lambda(i+j)+h)$. The reader may particle the expression $\sigma(\Omega(\mathcal{D}_4))$ as well as the initial states of $a^{-1}\alpha^{-1}\beta d$, $b^{-1}(bc^{-1}+de^{-1})c$, $a^{-1}\alpha^{-1}\beta(de^{-1}+de^{-1})\gamma^{-1}\delta f$ or $l(f(f(a,b),c))$.

5.2 Normalization and Confluence

Let G be a 2-graph. The subset $\Lambda_\sigma(G)$ is the closure under the PA transition relation of the subset $\sigma(\Gamma(G)) \subseteq St(G)$. The path P is a 2-path iff $P = C[Q_1 + \cdots + Q_n]$ implies $n = 2$.

Proposition 5.1 Normalization of PA. *On the state space $St(G)$ of a 2-graph G, the transition systems P and PA are well-founded, and confluent on 2-paths. If G is a 2-pograph, in the set $\Lambda_\sigma(G)$, the proper states of projective redexes are projective, non-trivial in the case of the product rule. The proper states in affine redexes are left or right affine contexts, possibly trivial and non-strict, the improper states in mixed redexes are non-trivial projective. In redexes, projective states are particle-free, and all particles of a proper affine state are displayed by its defining context. All particles in a redex are displayed in the corresponding left-hand side, and are moved by the transition that contracts this redex. In any state $\sigma \in \Lambda_\sigma(G)$, every source (resp. target) of σ possesses at least the negative (resp. positive) pebble that it had in $\sigma(p(\sigma))$, and at most one positive (resp. negative) pebble, the one it had in $\sigma(p(\sigma))$.*

Proof. The assertions on state types are proved by induction on transition sequences. They are true iff particles are typed: cf. $\Omega(\mathcal{D})$ with $\mathcal{D} \vdash_{LE}^{\{x=y\}} x = y$. The transition relation is well-founded. For a state σ in $St(G)$, we define $d(\sigma) = \sum_i d_\sigma(\varepsilon_i^-, \varepsilon_i^+)$ where $d_\sigma(\varepsilon_i^-, \varepsilon_i^+) = \max_k |A_k|$, if $A_k \in Sub_s(Q)$, Q the linearization of $p(\sigma)$, and $\varepsilon_i^- \in f^-(\partial^-(A_k))$ (resp. +). Then for any state σ we have $d(\sigma) \leq N \times |p(\sigma)|$, N its total number of particle pairs. But any transition $\sigma_1 \Rightarrow \sigma_2$ implies $d(\sigma_1) < d(\sigma_2)$. The relation is confluent on 2-paths, by the Newman and Knuth-Bendix lemmas: no projective rule uses contexts and the affine contexts are complete. Also by induction on transition sequences, every left-hand side exhibits all particles that are contained in its redexes. Hence there does not exist critical pairs between projective rules besides the confluent one between two instances of the exchange rule and the non-confluent self-critical pair of the sum rule. The positive (resp. negative) particle of a source (resp. target) particle pair is not duplicated until a terminal rule instance is applied to a particle set containing it. No projective particle is duplicated until a transition by a mixed rule. After a terminal rule instance, the moved particles are in source-target position and can no longer be displaced. Consequently, redexes are non-nested, i.e. no redex occurrence prefixes another redex occurrence; and we cannot have superpositions between affine and mixed rules. Nested redexes do exist if the type restriction is removed, cf. the example $\mathcal{E}_3$ and $\mathcal{D}_3$. From the completeness of contexts with respect to particles, there is no projective-proper affine superposition. Besides the non-confluent self-critical pair of the proper affine rules, involving paths that are not 2-paths, there only remains the superpositions between the exchange rule and the mixed rules. Their critical pairs are confluent after one transition. The one-transition step relation is confluent on 2-paths. $\square$

The section σ being given, proposition 5.1 defines a normal form section $\tau : \Gamma(G) \to St(G)$, which is defined on paths that are not 2-paths by arbitrarily choosing one of its irreducible states. An affine or mixed transition immediately followed by a projective one permute. From now on we consider only pebbling systems on P of the form $\sigma_0 = \sigma(P) \Rightarrow \cdots \Rightarrow \sigma_i \Rightarrow \cdots \Rightarrow \sigma_n$, where $\sigma_0 \Rightarrow \cdots \Rightarrow \sigma_i$ is purely projective and $\sigma_i \Rightarrow \cdots \Rightarrow \sigma_n$ is projective-transition free. By proposition 5.1, the state $\sigma_i = \rho(P)$ only depends on the 2-path P and defines a section of p, arbitrarily defined for the path P, when it is not a 2-path, to be one of its irreducible P-state such that $\rho(P) \Rightarrow \cdots \Rightarrow \tau(P)$.

Definition 5.1 *The path $P \in \Gamma(G)$ on the 2-pograph G is deadlock-free iff $\tau(P)$ is equal to $j(P)$: every source (resp. target) in $\tau(P)$ possesses every negative (resp. positive) particle. A path is extremal iff either it is non-trivial and its maximal sequential subpaths in $Sub_s, m(P)$ decompose in a product of braids or paths of the form $Q = a_1^{-1} \cdots a_n^{-1} \alpha b_1 \cdots b_m$, where a_i, b_j are atoms, $\partial^-(Q)$ and $\partial^+(Q)$ are target vertices and α is a source atom, or is an idempotent whose associated vertex is a target of G.*

Deduction semantics are extremal, but a deadlock-free connected path is not necessarily extremal, cf. the path $b^{-1}(bc^{-1} + de^{-1})c$ of the above example. The zero and non-variable idempotents are deadlock-free, but non extremal. Similarly, a non-zero product of two distinct non-target idempotents if deadlock-free via the terminal rule, but is not connected nor extremal.

Lemma 5.2 *The path semantics $\Omega(\mathcal{D})$ of the deduction $\mathcal{D} \vdash^{\mathcal{E}}_{LE} M = N$ is deadlock-free. Further, in transition sequences from $\sigma(\Omega(\mathcal{D})) \in \Lambda_\sigma(G)$, the proper states of product redexes are equality states. The proper states in sum redexes are projective. According to the types of its two projective states, instances of the exchange rule split up into introduction exchanges of type* (E,C) *or* (C,E), *transitivity exchanges of type* (EC,E) *or* (E,CE), *and connection exchanges of type* (CE,C) *or* (C,EC). *Any redex of any state σ derived from $\sigma(\Omega(\mathcal{D}))$ is associated to a unique inference rule instance or hypothesis in $\mathcal{D}$: sum redexes to binary rules with non-variable proper term, product redexes to eliminations, exchanges to binary rules or to the elimination rule of a phantom redex of the Cut equations* (19) – (22). *A left (right) affine redex σ is associated to an hypothesis iff $p(\sigma)$ is of the form $l(M)$, M some subterm occurrence in $\mathcal{E}$, and to a left introduction otherwise (resp. $r(M)$ and right introduction). The unique terminal redex in any particle system from $\sigma(\Omega(\mathcal{D}))$ is associated to the last inference of $\mathcal{D}$ and is the last transition in any system on $\Omega(\mathcal{D})$.*

Proof. By induction on deductions, the equality component E of any deduction $\mathcal{D}$ is deadlock-free, with $\rho(E) = \tau(E) = \varepsilon^- E \varepsilon^+$. The inference rule instance associated to a redex is the rule whose semantics include the subpath of the redex and whose distance to the sources is minimal. Its existence and uniqueness are proved by induction on deductions. The sole subtlety lies in the proof that if the redex is a product redex $a^{-1}\varepsilon^- E \varepsilon^+ b$, and $\Omega(\mathcal{D}') = L(E_1(R_1|L_1)E_2)$ for some subdeduction $\mathcal{D}'$ concluded by a transitivity rule, then it is impossible that the redex occurrence of a^{-1} belongs to E_1 while b's occurrence belongs to E_2. But this follows from the form of $\tau(E_i)$ and proposition 5.1. Notice that an exchange redex can be *internal* to a connection, created by an elimination, and in this latter case associated to this elimination in the case of cut equations (19)-(22).□
Let $\mathcal{D}$ be some deduction with $\Omega(\mathcal{D}) = L_1 E R_1$, we have:

$$\tau(\Omega(\mathcal{D})) = L_2[\varepsilon^-_{(N)}, \ldots, \varepsilon^-_{(N)}] E R_2[\varepsilon^+_{(N)}, \ldots, \varepsilon^+_{(N)}],$$

where (N) is the sequence $1, \ldots, N$, $((\varepsilon^-_i, \varepsilon^+_i))_{(N)}$ the sequence of particle pairs of $\sigma(\Omega(\mathcal{D}))$, and $p(L_2[\varepsilon^-_{(N)}, \ldots, \varepsilon^-_{(N)}]) = L_1$ (resp. R_1).

Let P be some deadlock-free path in $\Gamma(G)$, G a 2-graph, not necessarily projective. The product rule associates one pair of atoms, namely the atoms a and b in its definition; the sum rule two such pairs: the atoms a, b, and c, d in its definition. The particles in a set ε have *effectively travelled* through some atom a in $Sub(Q)$, Q the linearization of P, in a particle system on P iff, at some transition, the pair (ε, a) of the redex is:

- (ε^-, a) or (ε^+, b) of the product rule,

- (ε_1^-, a), (ε_1^+, b), (ε_2^-, c) or (ε_2^+, d) of the sum rule,
- (ε_1^+, c) or (ε_2^+, d) of the left affine rule,
- (ε_1^-, c) or (ε_2^-, d) of the right affine rule.

Now a particle has ***virtually travelled*** through some atom if it has effectively travelled through it or if its associated particle has virtually travelled through this occurrence, or if it has been associated to some particle, in a particle set, that has virtually travelled through it. With this definition, every particle has virtually visited any atom in the linearization of P, for any pebbling system on a deadlock-free path. Any atom in some deadlock-free path Q associated to P is effectively visited exactly once. Also, such an atom (recall that Q is linear) is *positive* (resp. *negative*) iff its associated particle as above is negative (resp. positive). In an equational setting, this definition is coherent with the usual definitions of positivity in proof-theory and with [35], §3.3. Atoms in the linear path associated to some path in $\Gamma(G)$ also split up into projective or affine ones, according to the type of the effective rule redex that possess this atom as proper atom, and possibly unspecified ones for non deadlock-free paths. Further, by the existence and uniqueness of normal forms, the proper atoms of effective redexes of some state in a particle system on P only depend on P, i.e. transition permutations preserve both the nature of the permuted rules and atom associations. An atom a of $Sub(Q)$, Q the linearization of a path $\Omega(\mathcal{D})$, may belong to several braids of $\Omega(\mathcal{D})$. However, with the notation of definition 5.1, the braid subpath $a_i^{-1}\cdots a_n^{-1}\alpha^{-1}\beta$ or $\alpha^{-1}\beta b_1\cdots b_j$, defined by $a = a_i^{-1}$ or $a = b_j$, is unique.

The local structure of deadlock-free, connected and extremal paths shows an *alternation* of target and source vertices along its sequential components, following the braid decomposition, reflecting the positive/negative partition of atoms.

Lemma 5.3 *A non-trivial deadlock-free connected path $P \in \Gamma(G)$ of a 2-pograph possesses a unique product decomposition $P = LER$ where L (resp. R) is a strict left (resp. right) affine path, and E a projective path. Conversely, such a connected product path is deadlock-free. Assume that G is the abstract graph of $T(\mathcal{E})$, for some set of equations $\mathcal{E}$, and that some atom associated to the equation $e : M_1 = M_2 \in \mathcal{E}$, whose source atom is α, belongs to $Sub(P)$, for a deadlock-free, connected and extremal path P. Then, the path P locally has the form of the semantics of some deduction, i.e. we may write:*

$$P = C[L[(P_1'|P_1),\ldots,(P_n'|P_n)](l(M_1)\backslash_l O_1)\alpha^{-1}\beta(r(M_2)\backslash_r O_2)R[(Q_1|Q_1'),\ldots,(Q_m|Q_m')]]$$

for some paths P_i', Q_j', $i = 1,\ldots,n$, $j = 1,\ldots,m$, where O_i is an occurrence of M_i, $i = 1,2$, and $O_1 \wedge O_2 = O_i$, $i = 1$ or 2, $M_1/O_1 \equiv D_1[S_1,\ldots,S_n]$, $M_2/O_2 \equiv D_2[T_1,\ldots,T_m]$, $L[\_,\ldots,\_] = l(D_1[\_,\ldots,\_])$, $R[\_,\ldots,\_] = r(D_2[\_,\ldots,\_])$, and $P_i = l(S_i)$, $Q_j = r(T_j)$, $i = 1,\ldots,n$, $j = 1,\ldots,m$. Further $O_1 \wedge O_2 = O_1$ (resp. O_2) implies $L[\_,\ldots,\_]$ trivial (resp. $R[\_,\ldots,\_]$).

Proof. First part of the lemma. By proposition 5.1, the path P is not of the form $P_1 + P_2$ for non-trivial paths P_1 and P_2. For, both P_1 and P_2 possess at least one source. And in the final state $\tau(P)$ one source cannot possess the negative particle of the other source. As this would occur by a move, instance of either the sum rule or a pure affine rule, e.g. if $\sigma(P)$ reduces to the left affine redex $A\varepsilon_1^+ a^{-1} + B\varepsilon_2^+ b^{-1}$, then, a^{-1} being a target of P_1, it possesses at least one positive particle by proposition 5.1, which is a contradiction. Hence $P = LR$, with at least one of L and R non-trivial. By the condition on particle types, the exchange, mixed and terminal rules cannot be interchanged. Also a positive source particle can reach a target only by an instance of the terminal rule. The analysis of paths in redexes conducted in proposition 5.1 concludes the existence of the LER decomposition. The uniqueness follows from the characterization of L as the subpath of P of the form $p(\sigma)$ where σ is the last left affine redex in any transition sequence from $\sigma(P)$ (resp. R). The converse is trivial by definition of projective and affine paths. Second part of the lemma. Let $P = C[a]$, a the given generator, and B be the unique braid subpath associated to the given occurrence of a, assumed positive, as defined before the lemma. This braid subpath exists, P being extremal and connected. If the effective rule associated to this occurrence of a is the product rule, either the occurrence of the generator b associated to a by this rule belongs to a braid including B or not. The only interesting case occurs when b does not belong to such a braid and is the first generator with this property among those associated by the product rule to a generator in B. For by proposition 5.1, a being positive, a subpath of the form $a(c^{-1}P_1 d)P_2 b$, with c, d associated, $P_2 = (Q_1|Q_1')P_3$ and $P_1 = l(M_1)\backslash_l O\alpha^{-1}\beta r(M_2)\backslash_r O$. In such a case $R[\_]$ is trivial. If a is effectively travelled by the sum rule, the subpath B is of the form $\alpha^{-1}\beta b_1 \cdots b_j$, with $a = b_j$, a's occurrence being positive. Consider the maximal path $b_k \cdots b_j$, $0 < k < j$, whose atoms are effectively travelled by a sum rule. This defines a Q_j occurrence in M_2, and a occurs in this Q_j. Similarly if the positive occurrence of a is travelled by a proper affine rule, a occurs in the context $L[\_, \ldots, \_]$. □

The semantics allows us to speak of auxiliary subderivations, in an intrinsic way: a deduction $\mathcal{D}'$ is a *left auxiliary subderivation* of $\mathcal{D}$ iff $\mathcal{D}'$ is a subdeduction of some deduction $\mathcal{D}''$, $\mathcal{D}'' =_{RE} \mathcal{D}$, $\Omega(\mathcal{D}') = L'E'R'$ and there exists a strict left affine path L'' such that $E'(R'|L'')$ is a maximal projective subpath of L, the left strict affine path of the LER decomposition of $\Omega(\mathcal{D})$ (resp. right). These subderivations are the auxiliary deductions of the normal form of the RE system in [35]. The set of principal hypotheses is linearly ordered. The leftmost and rightmost principal hypotheses are characterized on the semantics as follows. Every equality component is of the form $a^{-1}Eb$, a, b atoms. The left principal hypothesis (resp. right principal hypothesis) of $\mathcal{D}$ is the equation associated to a (resp. b) if $\Omega(\mathcal{D}) = L(a^{-1}Eb)R$, where L and R are respectively the left and right strict affine paths of $\Omega(\mathcal{D})$. The source particles that never are proper particles of a sum rule nor of a proper affine rule are called principal. In an equational setting, they semantically define the principal hypotheses of a deduction, notion which corresponds to the syntactical one of §4, by analysis of cut-free deductions. An hypothesis can be non-principal while its contribution belongs to the equality component of $\Omega(\mathcal{D})$.

5.3 Particle Systems on Space Graphs

Let now G be a 2-pograph. We examine the behaviour of particle systems on the graph $S(G)$, projected under the morphism $\pi : G \to S(G)$. One may define particle systems on the 2-graph $S(G)$, but these graphs are not circuit-free in general, hence there does no exist any natural initial state section of $p : St(S(G)) \to \Gamma(S(G))$. Notice that particles are not duplicated through projective transitions. Also, if P is some projective path, each state σ in $p^{-1}(P)$ defines a mapping v_σ from the particles in $\mathcal{P}(\sigma(P))$ to vertices of G.

Lemma 5.4 *Let P denote some deadlock-free projective path in $\Gamma(G)$ and $\pi : G \to S(G)$ the canonical projection. Associate atoms in P are identified by $\pi_\sharp$ in $\Gamma(S(G))$: eliminations and non-sequential binary rules conjugate projective paths by atoms in $\Gamma(S(G))$. Let $\sigma \in \Lambda_\sigma(G)$, we have $\pi(v_\sigma(\varepsilon_i^+)) = \pi(v_\sigma(\varepsilon_i^-))$ for any $\varepsilon_i^\pm \in \mathcal{P}(\sigma)$. Especially, $\partial_S^-(\pi_\sharp(P)) = \partial_S^+(\pi_\sharp(P))$ and $\pi_\sharp(P)$ is a closed projective path in $\Gamma(S(G))$. The projective path P is trivialized in $\Sigma(S(G))$: $\theta_T(P) \in \mathbb{N}^\star$, and $f_{S(G)} \circ \pi_\sharp(P)$ belongs to the subring $e_u\Delta(S(G))e_u$, where $u = \partial_S^\pm(\pi_\sharp(P))$. If the transition $\sigma \Rightarrow \rho$ contracts an exchange redex, we have $\pi(v_\sigma(\varepsilon_i^\pm)) = \pi(v_\rho(\varepsilon_i^\pm))$ for every particle $\varepsilon_i^\pm \in \mathcal{P}(\sigma)$.*

Proof. This follows from an easy induction on transition. $\square$

These results imply the existence of trajectories on $S(G)$ of particles. For a positive particle, this trajectory is the concatenation of the projections in $\Gamma(S(G))$ of the atoms associated to the edges effectively visited by the particle, in the order in which they are visited in any transition sequence. The trajectory of a negative particle is defined similarly. By lemma 5.4, these trajectories are independant of any specific transition sequence and are connected. The projective components of trajectories of associated projective particles are equal, modulo source atoms as usual. The trajectories are associated to transition rules that do not permute. Projective particles can also be partitionned according to their trajectories: either they include an affine component, or are purely projective. In an equational framework with $\mathcal{D} \vdash M = N$, the former are associated to auxiliary deductions, the latter to the principal subdeduction of $\mathcal{D}$. The affine component suffixes a trajectory and is of the form $a_1^{-1} \cdots a_n^{-1}$, a_i an atom, $\partial^-(a_n)$ a source, for positive particles; and of the form $b_1 \cdots b_m$, b_j an atom, $\partial^-(b_1)$ a source, for the negative ones. The trajectories of target particles, either affine or projective, never include source edges. A source (resp. target) affine particle has its trajectory equal to the image of a maximal branch of M (resp. N). This justifies the following interpretation of inference rules. To any inference is associated a vertex in $S(\mathcal{E})$: the common source of both terms of its conclusion. If we assume that any inference performs some projective operation, either at this vertex-point or somewhere else, then introductions are non-local while elimination and transitivity are local, i.e. operate at this vertex. In the dynamical system framework, the particles of a pair are not separated when travelling through an edge, *except* for source edges. Accordingly, the cut equations (19)-(22) may be called projective, which is coherent with the analysis of Cut-equations concluding §4.

6 Characterizations of Deduction Paths

Let $\mathcal{E}$ be some set of equations, such that $T(\mathcal{E})$ is connected. If $\mathcal{D} \vdash_{LE}^{\mathcal{E}} M = N$ is a deduction with $b \circ \partial^{-}(\Omega(\mathcal{D})) = \{u_i\}$ and $b \circ \partial^{+}(\Omega(\mathcal{D})) = \{v_j\}$. Then $\mathcal{D}$ encodes a two-way communication, firstly in $T(\mathcal{E})$, between the vertices u_i and the vertices v_j, hence, by projection, between the images of variable vertices under both morphisms $T(M) \to S(\mathcal{E})$ and $T(N) \to S(\mathcal{E})$. The particle game provides a first characterization of deduction semantics as deadlock-free, connected and extremal paths. From this first characterization, and by universal cover lifting, a second characterization of deduction paths is obtained, without resorting to pebbling. In turn, this last characterization can be used to get a variational interpretation of cut-elimination.

6.1 Full Graphs and Deductions Covering Graphs

We define the completions of both deductions and paths. Affine rules are disguised projective rules: in left (resp. right) affine rules, the negative (resp. positive) particles cannot move. A 2-graph G is full iff, for every vertex v of $S(G)$, there exists at least one target vertex u in G such that $\pi(u) = v$. Hence $G(\mathcal{E})$ is full iff for every vertex v of $S(\mathcal{E})$, there exists at least one variable $x \in \mathrm{O}_{S(\mathcal{E})}(v)$. Also $T(\mathcal{E})$ is embedded in one of its possible completions, the simplest one being $T(\mathcal{E}_\infty)$, where $\mathcal{E}_\infty = \mathcal{E} \cup \mathcal{H}(\mathcal{E})$ with $\mathcal{H}(\mathcal{E}) = \{x_{\infty,v} = M \mid M$ any $\mathcal{E}$-constructible term, $v = V_S(M)\}$, where $x_{\infty,v}$ is some distinguished variable. This completion has the following features:

1. $T(\mathcal{E}_\infty)$ possesses several target vertices ω_v "at infinity", with $\mathrm{O}_{T(\mathcal{E}_\infty)}(\omega_v) = \{x_{\infty,v}\}$.
2. For any $\mathcal{E}$-constructible term M, we have a unique injective equational morphism $T(M) \hookrightarrow T(\mathcal{H}(\mathcal{E}))$ with the vertex $V_{\mathcal{H}(\mathcal{E})}(M)$ equationally related to ω_v, $v = V_S(M)$.

If $\mathcal{D} \vdash_{LE}^{\mathcal{E}} M = N$ and $v = V_S(M) = V_S(N)$, its completion $\check{\mathcal{D}}$ is the deduction:

$$T\,\frac{x_{\infty,v} = M \qquad \overset{\mathcal{D}}{M = N} \qquad S\,\dfrac{x_{\infty,v} = N}{N = x_{\infty,v}}}{x_{\infty,v} = x_{\infty,v}}$$

where $x_{\infty,v} = M$ and $x_{\infty,v} = N$ belong to $\mathcal{H}(\mathcal{E})$. Similarly, any extremal connected deadlock-free path P can be completed via the connection operation as follows. By lemma 5.3, $P = LER$ with L strict left affine (resp. R right). To the path L is associated a unique simple path $R' = r(M)$ for the unique term occurrence of M in $\mathcal{H}(\mathcal{E})$ (resp. R, $L' = l(N)$).

We let $\check{P} = \alpha^{-1}(R'|L)E(R|L')\beta$, where α and β are the source atoms of the equations $x_{\infty,v} = M$ and $x_{\infty,v} = N$ respectively. Naturally $\check{\Omega}(\mathcal{D}) = \Omega(\check{\mathcal{D}})$, and $\check{P}$ is deadlock-free connected extremal: this follows from the permutation between projective and affine transitions.

In order to get a semantical explanation of the cut-elimination group, we introduce covering graphs of both the graph $T(\mathcal{E})$ and deductions. We assume that the path P is strongly connected deadlock-free extremal and projective. Hence it is a 1-1 path. Similarly, we assume that the deduction $\mathcal{D}$ is projective: $\mathcal{D} \vdash x = y$, $x, y \in \mathcal{V}$. We first describe the universal covering graph of $T(\mathcal{E})$ and its equational structure. The graph $\hat{T}(\mathcal{E})$ is defined as usual, e.g. through the standard topological representation (quotient of the product of n copies of the unit real interval, n the number of edges, by the vertex-edge incidence relation [49]). The graph $\hat{T}(\mathcal{E})$ is a tree and possess an equational structure, which make the projection $p : \hat{T}(\mathcal{E}) \to T(\mathcal{E})$ a geometric equational morphism. This structure is defined by $\mathcal{E}^{\infty} = \bigcup_{w \in \pi_1(T(\mathcal{E}))} Lin(\mathcal{E})_w$, where $Lin(\mathcal{E})$ is the linearization of $\mathcal{E}$ (cf. [35]), the disjoint sum being indexed by the first homotopy group of $T(\mathcal{E})$, two copies of $Lin(\mathcal{E})$ are related by the intersection $\mathcal{V}(Lin(\mathcal{E})_w) \cap \mathcal{V}(Lin(\mathcal{E})_{w'})$, which is a singleton when the element ww'^{-1} in $\pi_1(T(\mathcal{E}))$ is associated to a fundamental cycle, and empty otherwise. This is well-defined as the fundamental group of a graph is a free group.

The set of sequential closed subpaths that measure the parallelism present in the path P is defined by:

- $Sub_p(0) = Sub_p(a) = \emptyset$, if a is atomic;
- $Sub_p(P_1 + P_2) = Sub_p(P_1) \cup Sub_p(P_2)$;
- $Sub_p(P_1 P_2) = Sub_p(P_1) \cup \{cpc^{-1} \mid c$ some maximal path of $Sub_{s,m}(P_1)$, $p \in Sub_p(P_2)\}$.

Different choices for c give conjugate paths as $b \circ \partial^{+}(P_1) = b \circ \partial^{-}(P_2)$ is singleton, P being strongly connected. The group $H(P)$ is the normal subgroup of the fundamental group $\pi_1(G, v)$, with base point $\{v\} = b \circ \partial^{-}(P)$, generated by the homotopy classes of loops in $Sub_p(P)$. This group is independant of any choice for c. Let $\gamma_P : \hat{G}(P) \to G$ be the regular covering defined by $\gamma_{P\sharp}(\pi_1(\hat{G}(P))) = H(P)$. This covering graph is well-defined up to isomorphism by the classification theorem of covering spaces. If $P = \Omega(\mathcal{D})$, the graph $\hat{G}(P)$ possesses an equational structure, defined up to isomorphism, such that the map $\gamma_P : \hat{G}(P) \to T(\mathcal{E})$ becomes a geometric morphism of equation graphs. This is immediate as γ_P is topologically a local homeomorphism, and $\mathrm{O}_{T(\mathcal{E})}(v)$ is singleton when v is not a target vertex. By construction, this equational structure is quasi-linear.

Every vertex in $\hat{G}(P) = (V, E, \partial^{-}, \partial^{+})$ possesses a natural integer potential defined by:

- the potential of a base-point vertex is 0;

- if the potential $P(v)$ of a vertex v is k, and $v = \partial^+(\alpha)$, for some source edge α with complementary edge β, then $P(\partial^-(\alpha)) = P(\partial^+(\beta)) = k$;

- the potential $P(v)$ of a vertex v is $k+1$ if there exists a non-source edge e with $P(\partial^+(e)) = k$ and $v = \partial^-(e)$.

- the potential $P(v)$ is $k-1$ if there exists a non-source edge e with $P(\partial^-(e)) = k$ and $v = \partial^+(e)$.

This potential is well-defined as $P \in \Gamma(G)$ is projective connected. This is established as follows. By definition of $\hat{G}(P)$, any loop in $\pi_1(\hat{G}(P))$ has a decomposition $\prod_i x_i c_{i,1} c_{i,2}^{-1} x_i^{-1}$, where $c_{i,1}$ and $c_{i,2}$ are two sequential paths associated respectively to P_1 and P_2 of a subpath $P_1 + P_2$ in $Sub(P)$. But by unique decomposition, we have $P_1 + P_2 = aC_1c^{-1} + bC_2d^{-1}$, and say $c_{i,1}$ is associated to a maximal sequential path of aC_1c^{-1} (resp. $c_{i,2}$ to bC_2d^{-1}). As is easily seen, the potential of the source and of the target of a connected projective path are equal. Hence if two distinct sequential paths in $\hat{G}(P)$ have the same source and target, the potentials computed by either path are equal. Naturally, the path P can be lifted to $\hat{G}(P)$. This is established through any injection i of the graph $T(P)$ in $\hat{G}(P)$, which exists by definition of $\hat{G}(P)$. Further, when $P = \Omega(\mathcal{D})$, such an injection gives an equational isomorphism of $T(\mathcal{E}_\mathcal{D})$ onto its image in $\hat{G}(P)$, which gives an injection $S(T(\mathcal{E}_\mathcal{D})) \hookrightarrow S(\hat{G}(P))$. Hence we may define the lifting $\hat{P}$ of P to its cover $\hat{G}(P)$ as $i_\sharp(Q)$, where Q is the linearization of P.

6.2 Two Characterizations of Deduction Paths

We establish the converse of theorem 4.1: the existence and RE-unicity of a deduction associated to a deadlock-free connected and extremal path of $\Gamma(T(\mathcal{E}))$, $\mathcal{E}$ a set of equations. The particle systems provide a convenient way to achieve it. Without loss of generality, by projective completion and universal cover lifting, we may assume that the path P is projective and the deductions $\mathcal{D}_1$ and $\mathcal{D}_2$ linear with respect to the equations $\mathcal{E}$. Hence the path P is linearly expressed from the atoms of the path algebra. We first establish the uniqueness modulo RE of a deduction associated to P, by induction on the number of hypotheses. Assume $\mathcal{D}_1 \neq \mathcal{D}_2$ are both in RE-normal form, with $P = \Omega(\mathcal{D}_1) = \Omega(\mathcal{D}_2)$ projective. By uniqueness of the deadlock-free path decomposition, the left principal hypothesis of $\mathcal{D}_1$ is the left principal hypothesis of $\mathcal{D}_2$. Also let $\mathcal{D} \vdash M = N$ be the maximal proper subdeduction common to both $\mathcal{D}_1$ and $\mathcal{D}_2$, whose semantics contains this left principal hypothesis. We undertake a case analysis on the rules in $\mathcal{D}_1$ and $\mathcal{D}_2$ that possess $\mathcal{D}$'s conclusion $M = N$ as premiss. The branch in $\mathcal{D}_1$ or $\mathcal{D}_2$ from the equation $M = N$ to the conclusion is principal. Further, $\mathcal{D}$ leftmost and P projective exclude the cases where a subdeduction of $\mathcal{D}_1$ or $\mathcal{D}_2$, containing $\mathcal{D}$ as subdeduction is right premiss of a binary rule, and left premiss of a left introduction. This follows from the two observations that in the principal subdeduction of a normal form deduction, we do

not have any introduction in the leftmost principal branch, and that, here, these principal subdeductions are equal to $\mathcal{D}_1$ and $\mathcal{D}_2$, P being projective. There remains two cases.
(A) If $\mathcal{D}_1$ contains the subdeduction:

$$E\,\frac{\begin{matrix}\mathcal{D}\\ C_1[M_1] = D_1[M_1]\end{matrix}}{M_1 = N_1}$$

We establish that $\mathcal{D}$ cannot be the left premiss of a T-rule in $\mathcal{D}_2$. Let $A = (L\backslash_l O)E(R\backslash_r O)$ if $\Omega(\mathcal{D}) = LER$ and $O = O_{C_1} = O_{D_1}$. Contexts in rules being non-trivial, and by the observations following lemma 5.2, a projective atom is travelled by the same effective projective transition rule on any particle system with support P, and the associated inference rule is binary for the sum rule and an elimination for the product rule. Hence at least one atom in $R\backslash_r O$ would be associated to a sum rule in $\Omega(\mathcal{D}_2)$ and to a product rule in $\Omega(\mathcal{D}_1)$. This contradicts $\Omega(\mathcal{D}_1) \neq \Omega(\mathcal{D}_2)$. Similar arguments will be implicit in the rest of the proof. Finally, $\mathcal{D}_2$ contains the subdeduction:

$$E\,\frac{\begin{matrix}\mathcal{D}\\ C_2[M_2] = D_2[M_2]\end{matrix}}{M_2 = N_2}$$

The subdeduction $\mathcal{D}$ being chosen maximal, we have e.g. $C_1[\_] = C_2C_3[\_]$ and $D_1[\_] = D_2D_3[\_]$ for non-trivial contexts $C_3[\_]$ and $D_3[\_]$. Therefore the above elimination rule in $\mathcal{D}_2$ is not the concluding rule of this deduction. The case analysis above applies verbatim to the rule following this elimination in $\mathcal{D}_2$: either the semantics of $\mathcal{D}_1$ and $\mathcal{D}_2$ differ, or $\mathcal{D}_2$ contains a cut.
(B) Assume now that $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively contain the subdeductions $\mathcal{D}_1'$ and $\mathcal{D}_2'$:

$$T\,\frac{\begin{matrix}\mathcal{D}\\ M = N\end{matrix}\quad \begin{matrix}\mathcal{F}_1 & & \mathcal{F}_m\\ N = O_1 & \cdots O_{m-1} = & O_m\end{matrix}}{M = O_m} \qquad T\,\frac{\begin{matrix}\mathcal{D}\\ M = N\end{matrix}\quad \begin{matrix}\mathcal{G}_1 & & \mathcal{G}_n\\ N = P_1 & \cdots P_{n-1} = & P_n\end{matrix}}{M = P_n}$$

where the sequences of T-rules are maximal. From equations (7) and (10), both $\mathcal{F}_m$ and $\mathcal{G}_n$ are not concluded by a transitivity nor by a right introduction. The subdeduction $\mathcal{D}$ being leftmost in both $\mathcal{D}_1$ and $\mathcal{D}_2$, the deductions $\mathcal{D}_1'$ and $\mathcal{D}_2'$ are not left premiss of an introduction or a transitivity. Now, either both equality components of $\mathcal{D}_1'$ and $\mathcal{D}_2'$ are equal to the whole semantics P or not. In the former case, we apply the following observation. If the projective path E_1CE_2, product of the projective paths E_1 and E_2 with a connection path C, is deadlock-free, there exists two affine paths L and R such that $C = L|R$ and both E_1R and LE_2 are deadlock-free. Together with the uniqueness of the LER decomposition and the induction hypothesis, it implies the identity of $\mathcal{D}_1$ and $\mathcal{D}_2$. In the latter case, the sequence of transitions being maximal, the path P being projective and the equality components of both $\mathcal{D}_1'$ and $\mathcal{D}_2'$ being non maximal, there exists an elimination immediately below these subdeductions in both $\mathcal{D}_1$ and $\mathcal{D}_2$. The atoms occurrences

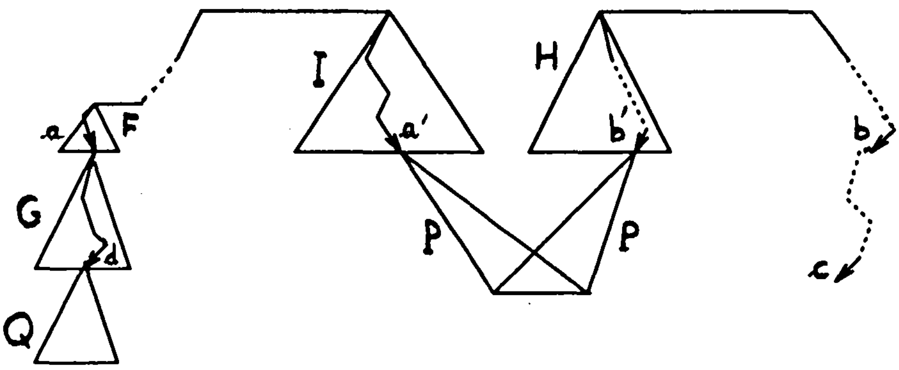

Figure 3: Equational surgery in the existence proof.

associated by these eliminations are the same in P, which implies $\Omega(\mathcal{D}'_1) = \Omega(\mathcal{D}'_2)$. And we conclude as above, by induction hypothesis.

The existence proof merges two adjacent "hypotheses" in some deadlock-free connected and extremal path, and uses an induction on the number n of equation occurrences in P, induction quantified over sets of equations. This is well-defined by the local structure of deduction paths, detailed in lemma 5.3. The cases $n = 1, 2$ are trivial. Let P be some linear projective extremal expression having n occurrences of equation semantics. The details of the surgery below can be followed on Fig. 3. We simply outline the general case.

Necessarily, there exist two "hypotheses" of P, $e_1 : M_1 = N_1$ and $e_2 : M_2 = N_2$, such that for an occurrence O_1 of N_1 (resp. O_2, M_2), we have $P \equiv N_1/O_1 \equiv M_2/O_2$ and:

$$P = C[a^{-1}E_1(r(N_1)\backslash_r O_1)][(r(N_1)/_r O_1)|(l(M_2)/_l O_2)](l(M_2)\backslash_l O_2)E_2 b],$$

where $r(N_1)\backslash_r O_1 = Aa'$, $l(M_2)\backslash_l O_2 = b'^{-1}B$, $a, b, a', b' \in \mathcal{G}$, and (a, a') and (b, b') are pairs of atomic paths associated by the product rule. The path P being deadlock-free, the expression $E_1 A$ is projective deadlock-free. Let $e_3 : M_3 = N_3$ be the unique equation whose graphical representation contains the edge a. By lemma 5.3, P being projective and $n > 2$, we can write:

$$P = D[(C|(l(M_3)/_l O_3))(l(M_3)\backslash_l O_3)\alpha_3^{-1}\beta_3(r(N_3)\backslash_r O'_3)E'_1(r(N_1)\backslash_r O_1)\cdots bE'_2 c],$$

where $l(M_3)\backslash_l O_3 = d^{-1}E_3 a^{-1}E_4$, $l(M_3)\backslash_l O''_3 = a^{-1}E_4$ and $E_4\alpha_3^{-1}\beta_3^{-1}(r(N_3)\backslash_r O'_3)E'_1 = E_1$. The pair (d, c) is associated by a product rule, the occurrences O_3, O''_3 are occurrences of M_3, O''_3 prefixing O_3. We write $M_3 \equiv FG[Q]$, with $O_{FG} = O_3$, $O_F = O''_3$, $M_2 \equiv H[P]$, with $O_H = O_2$, and $N_1 \equiv I[P]$, with $O_I = O_1$. We introduce the equations e_4 and e_5, where x is some new variable, $x \notin \mathcal{V}(\mathcal{E})$: $e_4 : HG[Q] = N_2$, $e_5 : F[x] = N_3$, which should be compared to $e_2 : H[P] = N_2$ and $e_3 : FG[Q] = N_3$. Let $\mathcal{E}'$ be the set of equations equal to $\mathcal{E}$ with e_2, e_3 replaced by e_4, e_5. By linearity of P, this substitution allows us to consider the two expressions:

$$
\begin{aligned}
E^1 &= a^{-1}E_1(r(N_1)\backslash_r O_1)(r(N_1)/_r O_1), \\
E^2 &= D[(C|l(Q))d^{-1}E_3(l(M_2)\backslash_l O_2)E_2 b E_2' c]
\end{aligned}
$$

in $\Gamma(\mathcal{E}')$, where atomic paths associated to e_2 and e_3 are unambiguously replaced by the atomic paths associated to e_4 and e_5. They are both deadlock-free, extremal, and their number of hypotheses is less than or equal to $n-1$. By induction hypothesis, there exist two deductions $\mathcal{D}_1$ and $\mathcal{D}_2$ with respective semantics E^1 and E^2. Further $\mathcal{D}_1 \vdash x = P$ and, by induction hypothesis, $\mathcal{D}_2$ contains the subdeduction $\mathcal{D}_3 \vdash Q = S$ for some term S as subdeduction. The situation is depicted below, where the branch from e_4 to $Q = S$, and the branches from e_5 to $x = P$ and from e_1 to $x = P$ are principal:

$$
\mathcal{D}_1 \left\{ \begin{array}{ccc} e_5 : F[x] = N_3 & \cdots & e_1 : M_1 = I[P] \\ & \vdots & \\ & x = P & \end{array} \right.
\qquad
\mathcal{D}_3 \left\{ \begin{array}{c} e_4 : HG[Q] = N_2 \\ \vdots \\ G[Q] = T \\ \vdots \\ Q = S \end{array} \right.
$$

Both E^1 and $\mathcal{E}'$ being linear with respect to the variable x, we may replace x by the term $G[Q]$, and built a labelled target tree $\mathcal{D}_4$ by gluing $\mathcal{D}_1$ and $\mathcal{D}_3$:

$$
\mathcal{D}_4 \left\{ \begin{array}{c} T \dfrac{\begin{array}{c} e_3 : FG[Q] = N_3 \quad \cdots \quad e_1 : M_1 = I[P] \\ \vdots \\ G[Q] = P \end{array} \qquad \begin{array}{c} e_2 : H[P] = N_2 \\ \vdots \\ P = S \end{array}}{G[Q] = S} \\ \vdots \\ Q = R \end{array} \right.
$$

We replace in $\mathcal{D}_2$ the subtree $\mathcal{D}_3$ by the tree $\mathcal{D}_4$. The resulting object is a deduction $\mathcal{D}$ from $\mathcal{E}$ whose semantics is P. Together with the fact that $\Omega(\mathcal{D}_x)$ is deadlock-free, connected and extremal, we have established the following:

Theorem 6.1 *Let $\mathcal{E}$ be some set of first-order equations. Let P be some path in $\Gamma(\mathcal{E})$. There exists a deduction $\mathcal{D} \vdash_{LE}^{\mathcal{E}} M = N$ such that $\Omega(\mathcal{D}) = P$ iff the path P is deadlock-free, connected and extremal. Further, $\mathcal{D}_1 =_{RE} \mathcal{D}_2$ iff $\Omega(\mathcal{D}_1) = \Omega(\mathcal{D}_2)$.*

A generalization of the covering graph $\hat{G}(P)$ for P connected non-projective is possible. In such a case, a path P is a denotation $\Omega(\mathcal{D})$ iff it is connected extremal and null-homotopic: the projection of $\hat{P}$ in $\Sigma(S(\hat{G}(P)))$ belongs to $\mathbf{N}^a st$.

6.3 A Discrete Variational Principle for Cut-free Deductions

Two natural strategies, a parallel one and a sequential one, are defined on deductions. They characterize respectively cut-free deductions and normal forms. A common feature of both strategies is the postponement of affine rules, according to the permutation result of §5. The parallel strategy proceeds from the hypotheses to the conclusion, and the sequential one follows a leftmost-outermost travelling of deduction trees.

The common part of affine transition sequences: left member of equalities are first played, leftmost redexes first, then right members, rightmost redexes first. Equality components being played, the mixed rules create affine redexes when needed. As usual, we only consider the projective behaviour of the systems.

The sequential strategy: when the redexes of an inference rule instance are contracted, the projective components of its premisses semantics are assumed to be contracted. A binary rule defines a transition sequence that include the sequence of the connection path of its proper term N. Let R_1 and L_2 be the paths associated to the two occurrences of N, as in the definition of deduction semantics in §3. All equality components of both R_1 and L_2 being reduced, the connection $(R_1|L_2)$ can be reduced in the outermost-leftmost strategy, also called the standard strategy [1], with 1) $|N|$ instances of the sum rule, where $|N|$ is the size of the term N, defined as the number of non-target vertices of its graphical representation; and 2) with exchange rules propagating the particles through equality components of R_1 and L_2. The transition sequence of the transitivity inference rule is the one of the connection $(R_1|L_2)$, followed by two exchanges.

$$T\,\frac{\overset{\mathcal{D}_1}{M=N} \quad \overset{\mathcal{D}_2}{N=O}}{M=O} \qquad \frac{(L_1,E_1,R_1) \quad (L_2,E_2,R_2)}{(L_1,E_1(R_1|L_2)E_2,R_2)}$$

The elimination rule is reduced with the product rule:

$$E\,\frac{\overset{\mathcal{D}}{C[M]=D[N]}}{M=N} \qquad \frac{(L,E,R)}{(L/_lO,(L\backslash_lO)E(R\backslash_rO),R/_rO)}$$

The product rule, applied $|O|$ times, plus some possible exchanges, reduces the equality component of the conclusion (equality components in $L\backslash_lO$ and $R\backslash_rO$ are reduced before the sequence of transitions specified by the product rule, by hypothesis). The projective transition sequence of introduction rules is similar to the transitivity rule sequence. For transitivity instances, the deduction $\mathcal{D}_1$ is projectively reduced, then $\mathcal{D}_2$. For introductions, the principal deduction is contracted before the other subdeductions, and finally the connection is reduced.

The groups *Ide*, *Seq*, *Par* and *Cut* have non-trivial interaction with particle systems. The *Can* group does not interact as the cancelled deduction is not taken into account in the semantics. And the *Sym* group is transparent with respect to pebbling. A semantical explanation of the set RE is provided by the analysis of equation (8):

$$T\,\frac{IR\,\dfrac{M=C[N] \quad N=O}{M=C[O]} \qquad C[O]=P}{M=P} \quad\Longrightarrow\quad T\,\frac{M=C[N] \qquad IL\,\dfrac{N=O \quad C[O]=P}{C[N]=P}}{M=P}$$

With the sequential strategy above, the transition sequence associated to the left-hand side of this equation does not follow a localization principle. The proper context $C[\_]$ of this equation being non-trivial, this particle system first moves particles of the left-most hypothesis equality component, then those of the equality component of the central hypothesis, which, $C[\_]$ being non-trivial, is nested in this path expression. Finally, it computes the equality component of the rightmost hypothesis and comes back to the semantics of the term O inside the semantics of the context $C[\_]$. This back and forth move disappears from the semantics of the right-hand side. As noticed in [35], this single choice of symmetry breaking determinates the overall structure of the system RE. If we restrict to sets of equations that respect the left-right symmetry of deductions, empirical evidence shows that there exists only two finite complete systems, namely the above set RE and the one described in [35].

The semantical analysis of the group Ide is now clear: besides the equation (8), the remaining equations select a leftmost first association, and a consequent ordering of transitions. The group Seq borrows its name from its action in reordering nested substitutions: in a parallel sequence of the deduction that follows a top-down strategy, these substitutions would be sequentially computed. Now the equations (11) and (12) are justified on the grounds that equality components are first computed. The localization principle is respected: we do not get out of the contex $D[\_]$, while computing inside the context $C[\_]$. The two equations (13) & (14) follows in a strict way the two principles: i) compute first the equality components, ii) according to a localization principle: see especially the numbering of the equality component α of equation (13). Notice that, in the group Seq, parenthesizing of left-hand sides and right-hand sides of equations (11) & (12) are equal, while they differ in equations (13) & (14). But the appearently distinct choices (left-parenthesizing in (13) versus right parenthesizing in (14)), are coherent with the alternation of variable and source vertices and particles. With respect to reduction strategies, the two systems RE of [35] and of the present paper merely differ by a different reduction of the projective components of the deduction paths. The RE-normal forms are characterized by the fact that their projective sequential transition sequence completely contracts a maximal equality component of their semantics, before passing to another such component.

The parallel strategy: we first observe that, given their common support $\Omega(\mathcal{D})$, from one transition system to another, an exchange redex is characterized through its product operator, e.g. $\varepsilon_1^- P_1 \varepsilon_1^+ . \varepsilon_2^- P_2 \varepsilon_2^+ P_3$ and $\varepsilon_1^- P_1 \varepsilon_1^+ . \varepsilon_2^- P_2 P_3 \varepsilon_2^+$, P_i projective, $i = 1,2,3$. Further, we distinguish left exchanges, of the form $((\ldots E).(C\ldots))$, and right ones, of the form $((\ldots C).(E\ldots))$, E and C an equality and a connection path, respectively. As noticed in the proof of lemma 5.2, there exist exchange redexes *internal* to connection subpaths, in the semantics of projective cut rules and of variable proper term rules. The other ones are called logical. We also remind from lemma 5.2 that to any exchange redex in some state inherited from $\sigma(\Omega(\mathcal{D}))$ is associated a unique inference rule instance or hypothesis of $\mathcal{D}$.

The principal equality component of a logical exchange redex associated to an inference rule instance in $\mathcal{D}$ is inductively defined on the subdeduction above this inference rule by:

- if the exchange is left and internal, or in an introduction, or in a transitivity rule, the component is the left equality component of the right premiss,
- if the exchange is right and internal, or in an introduction, or in a transitivity rule, the component is the right equality component of the left premiss,

The right equality component of an equation occurring in some deduction is inductively defined:

- for an hypothesis: it is the source atom;
- for the conclusion of a transitivity instance: it is the right equality component of the right premiss deduction;
- for the conclusion of both introduction instances: it is the right equality component of the principal premiss deduction,
- for the conclusion of an elimination instance: it is the equality component of the subdeduction including the elimination instance.

And symmetrically for the left equality component. If a left (resp. right) exchange redex is associated to a transitivity instance in $\mathcal{D}$, then it possesses an associated right (resp. left) exchange redex, as two exchange redexes logically characterize on deduction paths a transitivity instance. Such exchange redexes are called $\mathcal{D}$-conjugate. The parallel transition system of a deduction is then a sequence of successive states of the form:

$$\sigma_n \Rightarrow \sigma_n' \Rightarrow \sigma_n'' \Rightarrow \sigma_{n+1},$$

where are contracted in the first transition all structural redexes, in the second one all product redexes, and in the third one all exchange redexes, with the condition:

- structural redexes: between σ_n and σ_n' are contracted all redexes present in σ_n.
- product redexes: between σ_n' and σ_n'' is contracted any redex present in σ_n such that all maximal projective components, which belong to the semantics of the subdeduction whose last inference is the elimination associated to this product redex, are reduced in σ_n.
- exchange redexes: an exchange redex is contracted between σ_n'' and σ_{n+1} iff its $\mathcal{D}$-conjugate (when it exists) and itself are present in σ_n'', and its principal equality component as well as the component of its $\mathcal{D}$-conjugate (when it exists) are reduced in σ_n'.

Naturally, any transition system defined on $St(T(\mathcal{E}))$ can be lifted to $St(\hat{G}(P))$. And we may define, for a transition system $\sigma(\hat{E}) \Rightarrow \cdots \Rightarrow \tau(\hat{E})$ the sum over states σ''_n of the potentials of the vertices of the particles. A cut-free deduction minimizes this sum. This is established by first noticing that if $\mathcal{D}_1 =_{RE} \mathcal{D}_2$, the length of their parallel transition sequences are equal, second the sums for both left and right-hand side of the rules groups *Seq* and *Ide* are equal, while of course it strictly decreases by cut contraction.

It is possible to define the set of half-equations associated to some state σ, derived from $\sigma(\Omega(\mathcal{D}))$, $\mathcal{D} \vdash M = N$. When an exchange logical redex is contracted, this set is modifed, so that it converges towards the conclusion $M = N$. If we consider that the existence of $\mathcal{D}$-conjugate exchange redexes is a synchronization device, the rules (15) and (16) desynchronyze a deduction, while projective cuts synchronyze it. Finally, the following informal dynamical system description is useful: the graph $S(T(\mathcal{E}_\mathcal{D}))$ is a tree: $\mathcal{E}_\mathcal{D}$ is linear. Further we have $S(T(\mathcal{E}_\mathcal{D})) = S(\hat{G}(\Omega(\mathcal{D})))$ and the potential of by definition of the congruences $\sim$ and $\approx$. The particles of the auxiliary deductions of $\mathcal{D}$ converge towards the basin equal to the subtree associated to the linearization of $M = N$. The potential decreases from these auxiliary deductions towards this subtree, when the auxiliary subdeduction contains eliminations. The potential of particles, hence of the sum over states, increases under sum rules.

7 Homotopic Deductions, Sequentialization and Pruning

Having characterized deduction paths in the path algebras, we define some surgery operations needed in a classification of fixed-point deductions.

7.1 Homotopic Deductions

Let us first consider an example. The deduction $\mathcal{D}_8$, whose semantics is not homotopy minimal, is however homotopy irreducible as a deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{f(f(x,y),z) = f(f(a,b),c)}{f(x,y) = f(a,b)} \quad \dfrac{f(f(a,b),c) = f(f(x,y),z)}{f(a,b) = f(x,y)}}{f(x,y) = f(x,y)} \qquad f(x,y) = f(u,v)}{f(x,y) = f(u,v)}}{x = u}$$

With $T(\mathcal{E}_5)$ represented in Fig. 4 $\mathcal{E}_5 = \{f(f(x,y),z) = f(f(a,b),c),\ f(x,y) = f(u,v)\}$, we have $\Omega(\mathcal{D}_5) = e^{-1}g^{-1}\beta i(kk^{-1} + ll^{-1})i^{-1}\beta^{-1}g(ec^{-1} + fd^{-1})\alpha^{-1}a$, whose contraction $e^{-1}(ec^{-1} + fd^{-1})\alpha^{-1}a$ does not look like any deduction semantics. This expression is connected, deadlock-free but non-extremal. It is quite important however to notice that

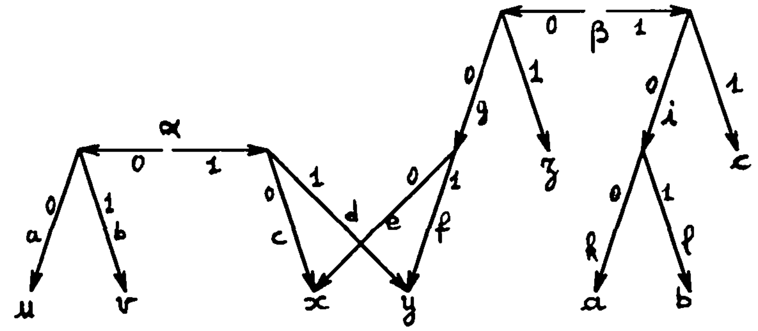

Figure 4: The graph $T(\mathcal{E}_5)$

the sequentialized and pruned form of $\mathcal{D}_5$ *is* homotopy contractible:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{f(f(x,y),z)=f(f(a,b),c)}{f(x,y)=f(a,b)} \quad \dfrac{\dfrac{f(f(a,b),c)=f(f(x,y),z)}{f(a,b)=f(x,y)}}{a=x}}{f(x,y)=f(x,b)}}{x=x} \qquad f(x,y)=f(u,v)}{f(x,y)=f(u,v)}}{x=u}$$

has semantics $e^{-1}g^{-1}\beta ikk^{-1}i^{-1}\beta^{-1}gec^{-1}\alpha^{-1}a$, whose contraction $c^{-1}\alpha^{-1}a$ is the semantics of the deduction:

$$\dfrac{f(x,y)=f(u,v)}{x=u}$$

Theorem 7.1 *Let $\mathcal{E}$ be a set of first-order equations and $\mathcal{D}\vdash^{\mathcal{E}}_{LE} M=N$ be some deduction. There exists a unique deduction $\mathcal{D}'\vdash^{\mathcal{E}}_{LE} M=N$ whose semantics is the homotopy contraction of $\Omega(\mathcal{D})$.*

Proof. By the alternation property, we define an homotopy redex associated to some occurrence in $\Omega(\mathcal{D})$ of the semantics of the equation $e: M=N$ from $\mathcal{E}$, $M\equiv C[P]$, $N\equiv D[Q]$ for some, possibly trivial, equivalent contexts $C[\_]$ and $D[\_]$, to be a subexpression of the form:

$$(A|l(P))(l(M)\backslash_l O)\alpha^{-1}\beta(r(N)\backslash_r O)[r(Q)|l(Q)](l(N)\backslash_l O)\beta^{-1}\alpha(r(M)\backslash_r O)(r(P)|B),$$

where $O=O_C=O_D$, whose contraction is $A|B$. The multipath $\Omega(\mathcal{D})$ being deadlock-free, a contraction of some homotopy redex in $\Omega(\mathcal{D})$ yields a deadlock-free extremal path E. By theorem 6.1, we get some deduction $\mathcal{D}'$ such that $\Omega(\mathcal{D}')=E$. Further, the contraction erases a projective subexpression of $\Omega(\mathcal{D})$, hence $\mathcal{D}\vdash^{\mathcal{E}}_{LE} M=N$. Finally, this reduction is trivially confluent and well-founded, possibly yielding $\mathcal{D}_x$. □

This theorem introduces the notion of null-homotopic deduction, of the form $\mathcal{D}\vdash M=M$, with $\Omega(\mathcal{D})=A.E.A^{-1}$, the equality component E being null-homotopic, modulo the source atoms, in $S(\mathcal{E})$ as is every equality component.

7.2 Sequentialization and Pruning

We conclude the deduction surgery by defining both the sequentialization and the pruning of a deduction. The pruning operation is quite natural: we remove unneeded branches in some deduction. It has already been introduced by Goad in a natural deduction setting [19]. In our framework, assume given some variable occurrence O of say N if $\mathcal{D} \vdash M = N$. Let $\Omega(\mathcal{D}) = D[(R'|L)ER]$, where $LER = \Omega(\mathcal{D}')$ and $\mathcal{D}'$ is an auxiliary subdeduction of $\mathcal{D}$ whose conclusion $O = P$ is such that $N = C[P]$ and O,O_C are disjoint: say $O = O_1 0 O_2$ and $O_C = O_1 1 O_3$, then we merely remove from $\mathcal{D}'$ this subdeduction to get a new deduction $\mathcal{D}'' \vdash M = C[O]$, with $\Omega(\mathcal{D}'') = D[R']$. A deduction is sequential iff all its equality or projective components are sum-free. This is a semantical definition, but not homotopy invariant of course. Its origin lies in the type inference problem: a necessary condition for some algorithm to possess some higher-order typing is that its first-order obstruction circuits are resolved. These obstruction circuits are defined by fixed-point deductions, and it is intuitively clear that protecting the sequential fixed-point deductions will protect the non-sequential ones as well. For a correct understanding of sequentialization, one should keep in mind that two deductions conjugate along some circuit of $S(\mathcal{E})$ have the same global operational behaviour. At this point, it is clear that sequentialization, contrarily to homotopy contraction, does not preserve the conclusions of deductions: if $\mathcal{D} \vdash^{\mathcal{E}}_{LE} M = N$, then we are searching some naturally associated deductions in LE_s, but it may well be the case that not $\models^{\mathcal{E}}_{U} M = N$, or even worse that $\models^{\mathcal{E}}_{U} M = N$, but not $\vdash^{\mathcal{E}}_{LE_s} M = N$. In particular, if $\mathcal{D} \vdash^{\mathcal{E}}_{LE} x = C[x]$, then $V_U(x)$ may be non-cyclic in $U(\mathcal{E})$, where deducibility in LE_s should be understood.

We give the sequentialization contractions. Notice that they are related, in the affine case, to the distributiviy law. As usual, lower-case letters denote atoms, P, P_i and C_i, $i = 1,2$, projective paths, L_i and R_i, $i = 1,2$, respectively left and right affine paths, possibly non-strict. The affine ones:

$$(aC_1c^{-1} + bC_2d^{-1})P(eR_1 + fR_2) \Rightarrow a(C_1(c^{-1}Pe))R_1 + b(C_2(d^{-1}Pf)R_2$$

$$(L_1a^{-1} + L_2b^{-1})P(cC_1d^{-1} + eC_2f^{-1}) \Rightarrow L_1(a^{-1}Pc)C_1d^{-1} + L_2(b^{-1}Pe)C_2f^{-1}$$

Notice the affine-projective shift from the left-hand side to the right-hand side. The projective sequentialization contractions:

$$a^{-1}P_1(bC_1c^{-1} + dC_2e^{-1})P_2f \Rightarrow (a^{-1}P_1b)C_1(c^{-1}P_2f) \quad \text{if } l(a) = l(b);$$

$$a^{-1}P_1(bC_1c^{-1} + dC_2e^{-1})P_2f \Rightarrow (a^{-1}P_1d)C_1(e^{-1}P_2f) \quad \text{if } l(a) = l(d);$$

They are non-confluent, but clearly normalizing. The right-hand sides are deadlock-free connected and extremal iff the left-hand sides are. They are performed on fixed-point deduction semantics, up to a cyclic permutation of hypotheses.

Theorem 7.2 *To any fixed-point deduction $\mathcal{D} \vdash^{\mathcal{E}}_{LE} x = C[x]$ is canonically associated an element of $\pi_1(T(\mathcal{E}))$, the fundamental group of the graphical representation of $\mathcal{E}$.*

Proof. By combining the homotopy, pruning, sequential and cyclic permutation surgery on the semantics of a fixed-point deduction. These operations are performed without leaving the set of deduction denotations. □

We conclude by considering the extension to arbitrary operator domains of the above material. In the definition of the equivalences $\sim$ and $\approx$, one should add the condition that if two edges are equivalent, their labelling operators should be equal. Further, the equivalence $\approx$ should be locally $\sim$-closed: cf. the example of an equational graph such that two distinct vertices contain respectively the sets of terms $\{f(a,b),\ g(u,g(v,w))\}$ and $\{f(a,b),\ g(u,v)\}$. We then consider *arity-preserving* paths on equational graphs, which are connected paths P of the form $P = LER$, with the additional constraint that if $R = C[\sum_{1\leq i\leq n} a_i P_i R_i]$, the common source of the a_i is a vertex of G with outdegree n (resp. L, E). The PA-sum rules, both affine and projective, can be easily modified so as to preserve deadlock-freeness of deductions semantics: it suffices to add an arity condition on edges, besides their labelling. The definition of the group $H(P)$, P some path, should also be modified consequently.

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B.P. 105, 78153 Le Chesnay Cedex, France.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique.

ISSN 0249-6399